دانشکده مهندسی مکانیک

# Chapter 4: Kinematics of Rigid Bodies

## Advanced Dynamics

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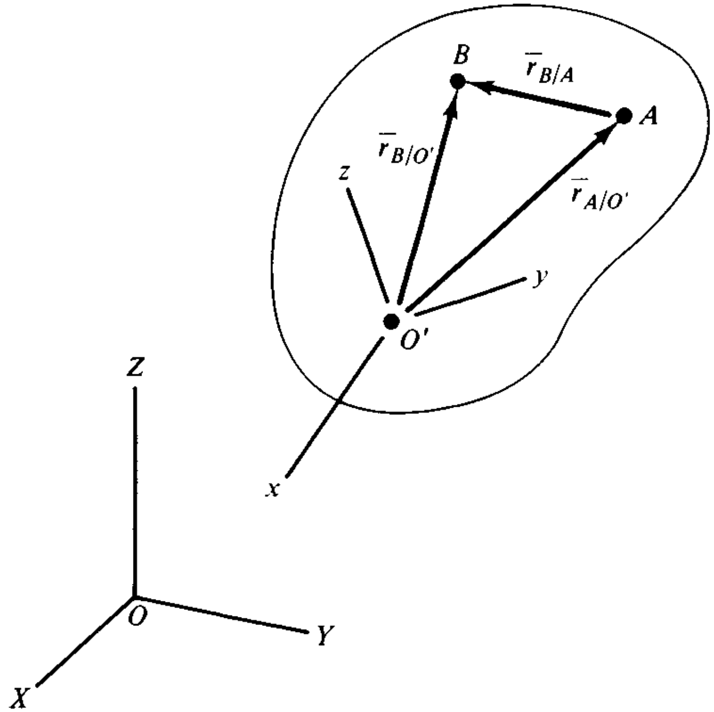
- A **rigid body** is defined to be a collection of particles whose distance of separation is invariant. In this circumstance, any set of coordinate axes xyz that is scribed in the body will maintain its orientation relative to the body. Such a coordinate system forms a ***body-fixed reference frame***. The orientation of xyz relative to the body and the location of its origin are arbitrary.

- The velocity and acceleration of point A

on the body are given by:

$$\bar{v}_A = \bar{v}_{O'} + \bar{\omega} \times \bar{r}_{A/O'},$$

$$\bar{a}_A = \bar{a}_{O'} + \bar{\alpha} \times \bar{r}_{A/O'} + \bar{\omega} \times (\bar{\omega} \times \bar{r}_{A/O'}).$$



- For another point such as B, we have similar relations as:

$$\bar{v}_B = \bar{v}_{O'} + \bar{\omega} \times \bar{r}_{B/O'},$$

$$\bar{a}_B = \bar{a}_{O'} + \bar{\alpha} \times \bar{r}_{B/O'} + \bar{\omega} \times (\bar{\omega} \times \bar{r}_{B/O'}).$$

- Using the previous relations for points A and B, we will have:

$$\bar{v}_B - \bar{v}_A = \omega \times (\bar{r}_{B/O'} - \bar{r}_{A/O'}),$$

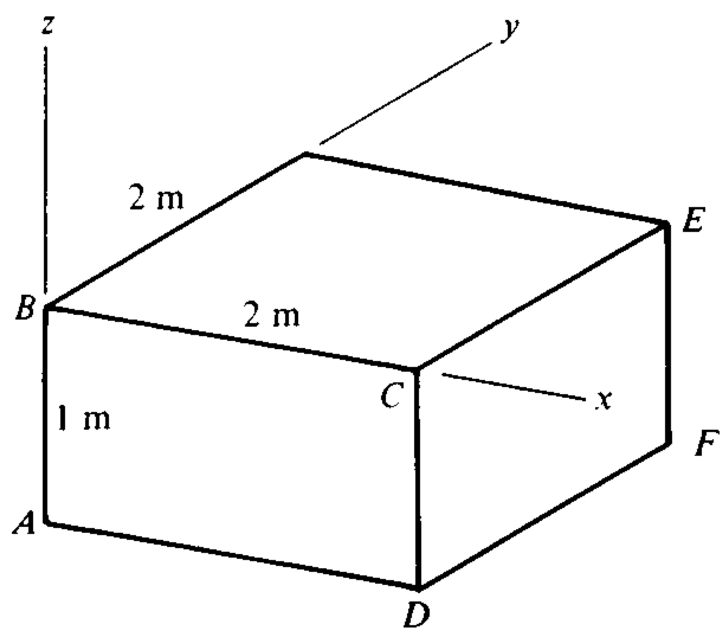
$$\bar{a}_B - \bar{a}_A = \bar{\alpha} \times (\bar{r}_{B/O'} - \bar{r}_{A/O'}) + \bar{\omega} \times [\bar{\omega} \times (\bar{r}_{B/O'} - \bar{r}_{A/O'})].$$

- Because $\bar{r}_{B/A} = \bar{r}_{B/O'} - \bar{r}_{A/O'}$, we find that:

$$\bar{v}_B = \bar{v}_A + \bar{\omega} \times \bar{r}_{B/A},$$

$$\bar{a}_B = \bar{a}_A + \bar{\alpha} \times \bar{r}_{B/A} + \bar{\omega} \times (\bar{\omega} \times \bar{r}_{B/A}).$$

- **Example:** Observation of the motion of the block reveals that at a certain instant the velocity of corner A is parallel to the diagonal AE. At this instant components relative to the body-fixed xyz coordinate system of the velocities of the other corners are known to be $(v_B)_x = 10$, $(v_c)_z = 20$, $(v_D)_x = 10$, and $(v_E)_y = 5$, where all values are in units of meters/second. Determine whether these values are possible, and if so, evaluate the velocity of corner F.



# 4.2 EULERIAN ANGLES

- Three independent direction angles define the orientation of a set of xyz axes. Eulerian angles treat this matter as a specific sequence of rotations.

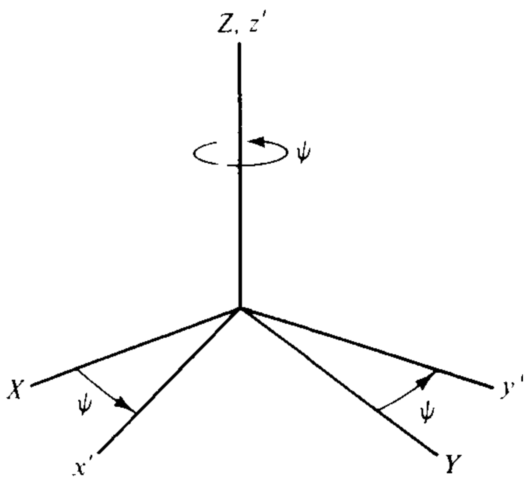



**Figure 4.2** Precession.

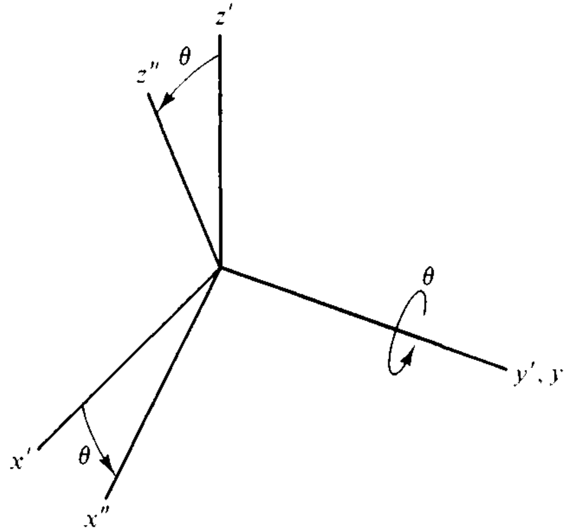



**Figure 4.3** Nutation.

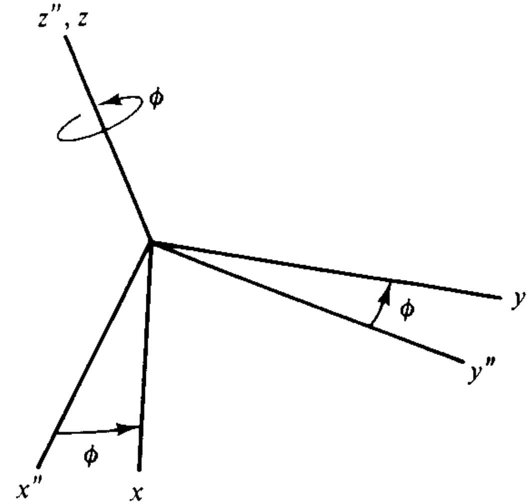
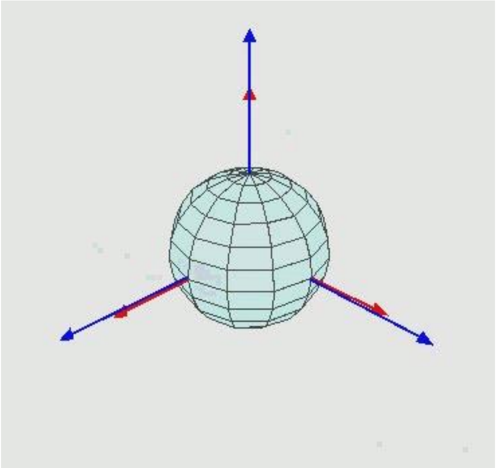
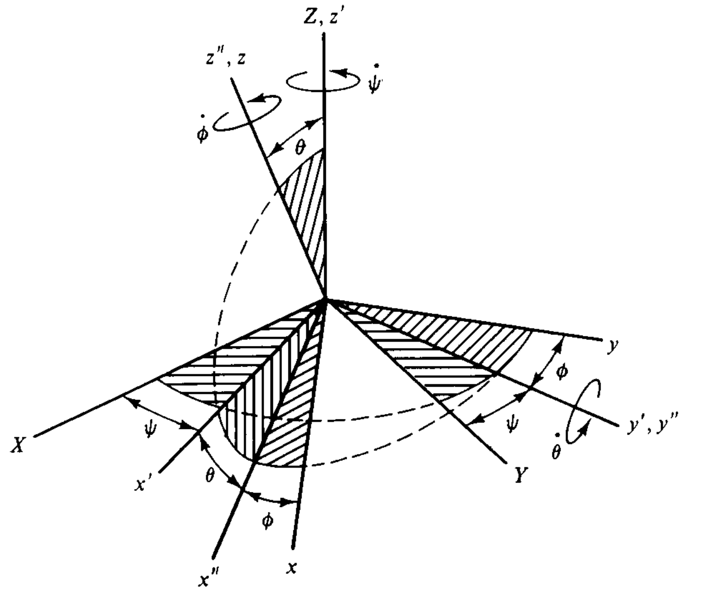



**Figure 4.4** Spin.





- The transformation from $XYZ$ to $x'y'z'$ may be found from to be

$$\begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix} = [R_\psi] \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix}, \qquad [R_\psi] = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

- The second transformation is given by

$$\begin{Bmatrix} x'' \\ y'' \\ z'' \end{Bmatrix} = [R_\theta] \begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix}, \qquad [R_\theta] = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}.$$

- The last transformation is given by

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = [R_\phi] \begin{Bmatrix} x'' \\ y'' \\ z'' \end{Bmatrix}, \qquad [R_\phi] = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

- The transformation matrix from first reference frame to the last one is given by:

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = [R] \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix}, \quad [R] = [R_\phi][R_\theta][R_\psi].$$

- The angular velocity and angular acceleration are readily expressed in terms of the angles of precession, nutation, and spin by adding the rotation rates about the respective axes. The angular velocity of xyz is the (vector) sum of the individual rotation rates, so

$$\bar{\omega} = \dot{\psi}\bar{K} + \dot{\theta}\bar{j}' + \dot{\phi}\bar{k}.$$

- The angular velocity of x'y'z' reference frame is:

$$\bar{\omega}' = \dot{\psi}\bar{K}.$$

- We can use the expressions for $\boldsymbol{\omega}$ and $\boldsymbol{\omega}'$ to obtain angular acceleration as:

$$\begin{aligned}\bar{\alpha} &= \ddot{\psi}\bar{K}+\ddot{\theta}\bar{j}'+\dot{\theta}\dot{\bar{j}}'+\ddot{\phi}\bar{k}+\dot{\phi}\dot{\bar{k}}\\ &= \ddot{\psi}\bar{K}+\ddot{\theta}\bar{j}'+\dot{\theta}(\bar{\omega}'\times\bar{j}')+\ddot{\phi}\bar{k}+\dot{\phi}(\bar{\omega}\times\bar{k}).\end{aligned}$$

- Expressions for unit vectors in terms of body coordinate vector are:

$$\begin{aligned}\bar{K} &= \sin\theta[-(\cos\phi)\bar{i}+(\sin\phi)\bar{j}]+(\cos\theta)\bar{k},\\ \bar{j}' &= (\sin\phi)\bar{i}+(\cos\phi)\bar{j}.\end{aligned}$$

- Thus, the angular velocity and angular acceleration are

$$\begin{aligned}\bar{\omega} = {} & (-\dot{\psi}\sin\theta\cos\phi+\dot{\theta}\sin\phi)\bar{i}\\ & +(\dot{\psi}\sin\theta\sin\phi+\dot{\theta}\cos\phi)\bar{j}+(\dot{\psi}\cos\theta+\dot{\phi})\bar{k},\end{aligned}$$

$$\begin{aligned}\bar{\alpha} = {} & (-\ddot{\psi}\sin\theta\cos\phi+\ddot{\theta}\sin\phi-\dot{\psi}\dot{\theta}\cos\theta\cos\phi+\dot{\psi}\dot{\phi}\sin\theta\sin\phi+\dot{\phi}\dot{\theta}\cos\phi)\bar{i}\\ & +(\ddot{\psi}\sin\theta\sin\phi+\ddot{\theta}\cos\phi+\dot{\psi}\dot{\theta}\cos\theta\sin\phi+\dot{\psi}\dot{\phi}\sin\theta\cos\phi-\dot{\theta}\dot{\phi}\sin\phi)\bar{j}\\ & +(\ddot{\psi}\cos\theta+\ddot{\phi}-\dot{\psi}\dot{\theta}\sin\theta)\bar{k}.\end{aligned}$$

- These expressions, particularly the one for angular aceleration, are quite complicated. For that reason, the x"y"z" axes, which do not undergo the spin, are sometimes selected for the representation. Then

$$\bar{K} = -(\sin\theta)\bar{i}'' + (\cos\theta)\bar{k}''; \qquad \bar{j}' = \bar{j}'', \qquad \bar{k} = \bar{k}''.$$

- The new expressions for angular velocity and expressions can be derived using those previous formulas by setting $\phi = 0$.

- Rotation matrix for different sequences selections:

| Proper Euler angles | Tait-Bryan angles |
|---|---|
| $X_1Z_2X_3 = \begin{bmatrix} c_2 & -c_3s_2 & s_2s_3 \\ c_1s_2 & c_1c_2c_3 - s_1s_3 & -c_3s_1 - c_1c_2s_3 \\ s_1s_2 & c_1s_3 + c_2c_3s_1 & c_1c_3 - c_2s_1s_3 \end{bmatrix}$ | $X_1Z_2Y_3 = \begin{bmatrix} c_2c_3 & -s_2 & c_2s_3 \\ s_1s_3 + c_1c_3s_2 & c_1c_2 & c_1s_2s_3 - c_3s_1 \\ c_3s_1s_2 - c_1s_3 & c_2s_1 & c_1c_3 + s_1s_2s_3 \end{bmatrix}$ |
| $X_1Y_2X_3 = \begin{bmatrix} c_2 & s_2s_3 & c_3s_2 \\ s_1s_2 & c_1c_3 - c_2s_1s_3 & -c_1s_3 - c_2c_3s_1 \\ -c_1s_2 & c_3s_1 + c_1c_2s_3 & c_1c_2c_3 - s_1s_3 \end{bmatrix}$ | $X_1Y_2Z_3 = \begin{bmatrix} c_2c_3 & -c_2s_3 & s_2 \\ c_1s_3 + c_3s_1s_2 & c_1c_3 - s_1s_2s_3 & -c_2s_1 \\ s_1s_3 - c_1c_3s_2 & c_3s_1 + c_1s_2s_3 & c_1c_2 \end{bmatrix}$ |
| $Y_1X_2Y_3 = \begin{bmatrix} c_1c_3 - c_2s_1s_3 & s_1s_2 & c_1s_3 + c_2c_3s_1 \\ s_2s_3 & c_2 & -c_3s_2 \\ -c_3s_1 - c_1c_2s_3 & c_1s_2 & c_1c_2c_3 - s_1s_3 \end{bmatrix}$ | $Y_1X_2Z_3 = \begin{bmatrix} c_1c_3 + s_1s_2s_3 & c_3s_1s_2 - c_1s_3 & c_2s_1 \\ c_2s_3 & c_2c_3 & -s_2 \\ c_1s_2s_3 - c_3s_1 & s_1s_3 + c_1c_3s_2 & c_1c_2 \end{bmatrix}$ |
| $Y_1Z_2Y_3 = \begin{bmatrix} c_1c_2c_3 - s_1s_3 & -c_1s_2 & c_3s_1 + c_1c_2s_3 \\ c_3s_2 & c_2 & s_2s_3 \\ -c_1s_3 - c_2c_3s_1 & s_1s_2 & c_1c_3 - c_2s_1s_3 \end{bmatrix}$ | $Y_1Z_2X_3 = \begin{bmatrix} c_1c_2 & s_1s_3 - c_1c_3s_2 & c_3s_1 + c_1s_2s_3 \\ s_2 & c_2c_3 & -c_2s_3 \\ -c_2s_1 & c_1s_3 + c_3s_1s_2 & c_1c_3 - s_1s_2s_3 \end{bmatrix}$ |
| $Z_1Y_2Z_3 = \begin{bmatrix} c_1c_2c_3 - s_1s_3 & -c_3s_1 - c_1c_2s_3 & c_1s_2 \\ c_1s_3 + c_2c_3s_1 & c_1c_3 - c_2s_1s_3 & s_1s_2 \\ -c_3s_2 & s_2s_3 & c_2 \end{bmatrix}$ | $Z_1Y_2X_3 = \begin{bmatrix} c_1c_2 & c_1s_2s_3 - c_3s_1 & s_1s_3 + c_1c_3s_2 \\ c_2s_1 & c_1c_3 + s_1s_2s_3 & c_3s_1s_2 - c_1s_3 \\ -s_2 & c_2s_3 & c_2c_3 \end{bmatrix}$ |
| $Z_1X_2Z_3 = \begin{bmatrix} c_1c_3 - c_2s_1s_3 & -c_1s_3 - c_2c_3s_1 & s_1s_2 \\ c_3s_1 + c_1c_2s_3 & c_1c_2c_3 - s_1s_3 & -c_1s_2 \\ s_2s_3 & c_3s_2 & c_2 \end{bmatrix}$ | $Z_1X_2Y_3 = \begin{bmatrix} c_1c_3 - s_1s_2s_3 & -c_2s_1 & c_1s_3 + c_3s_1s_2 \\ c_3s_1 + c_1s_2s_3 & c_1c_2 & s_1s_3 - c_1c_3s_2 \\ -c_2s_3 & s_2 & c_2c_3 \end{bmatrix}$ |

# QUATERNIONS

- The quaternion's basic definition is a consequence of the properties of the direction cosine matrix [A]. It is shown by linear algebra that a proper real orthogonal 3 X 3 matrix has at least one eigenvector with eigenvalue of unity.
- The quaternion is defined as a vector by Hamilton 1866, Goldstein 1950, and Dalquist 1990.
- The elements of the quaternions, sometimes called the *Euler symmetric parameters*, can be expressed in terms of the principal eigenvector e (see Sabroff et al. 1965). They are defined as follows:

$$\begin{aligned} q_1 &= e_1 \sin(\alpha/2), \\ q_2 &= e_2 \sin(\alpha/2), \\ q_3 &= e_3 \sin(\alpha/2), \\ q_4 &= \cos(\alpha/2). \end{aligned}$$

$$q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1, \qquad |\mathbf{q}| = 1.$$

- The rotation matrix in terms of quaternions are:

$$[\mathbf{A}(\mathbf{q})] = \begin{bmatrix} q_1^2 - q_2^2 - q_3^2 + q_4^2 & 2(q_1q_2 + q_3q_4) & 2(q_1q_3 - q_2q_4) \\ 2(q_1q_2 - q_3q_4) & -q_1^2 + q_2^2 - q_3^2 + q_4^2 & 2(q_2q_3 + q_1q_4) \\ 2(q_1q_3 + q_2q_4) & 2(q_2q_3 - q_1q_4) & -q_1^2 - q_2^2 + q_3^2 + q_4^2 \end{bmatrix}.$$

- **Time Derivation of the Quaternion Vector:**

As with the previous case, a differential vector equation for q can be written if the angular velocity vector of the body frame is known with respect to another reference frame. The differential equations of the quaternion system become

$$\frac{d}{dt}\mathbf{q} = \frac{1}{2}[\boldsymbol{\Omega}']\mathbf{q}, \qquad [\boldsymbol{\Omega}'] = \begin{bmatrix} 0 & \omega_z & -\omega_y & \omega_x \\ -\omega_z & 0 & \omega_x & \omega_y \\ \omega_y & -\omega_x & 0 & \omega_z \\ -\omega_x & -\omega_y & -\omega_z & 0 \end{bmatrix}.$$

$$v' = \begin{bmatrix} v_1' \\ v_2' \\ v_3' \end{bmatrix} = \begin{bmatrix} (1-2q_2^2-2q_3^2) & 2(q_1q_2+q_0q_3) & 2(q_1q_3-q_0q_2) \\ 2(q_1q_2-q_0q_3) & (1-2q_1^2-2q_3^2) & 2(q_2q_3+q_0q_1) \\ 2(q_1q_3+q_0q_2) & 2(q_2q_3-q_0q_1) & (1-2q_1^2-2q_2^2) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$\begin{bmatrix} \dot{q}_0 \\ \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & -p & -q & -r \\ p & 0 & r & -q \\ q & -r & 0 & p \\ r & q & -p & 0 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} + K\varepsilon \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

$$\varepsilon = 1 - ({q_0}^2 + {q_1}^2 + {q_2}^2 + {q_3}^2)$$

- **A Comparison Between quaternion and Euler angles in the field of speed**

```
clc
clear alle
close all

q0=[0 0 0 1]';
angles0=[0 0 0]';

TimeSpan=[0 10000];


t0 = tic;
[t,q]=ode45(@quat,TimeSpan,q0);
toc(t0)

t2 = tic;
[t1,angles]=ode45(@Euler,TimeSpan,an
gles0);
toc(t2)
```

| Simulation Time | Elapsed Time (Quaternion Method) | Elapsed Time (Euler Method) |
|---|---|---|
| 1000 | 1.88 | 3.96 |
| 2000 | 2.64 | 8.84 |
| 5000 | 4.46 | 22.8 |
| 10000 | 5.80 | 46.4 |

$$\dot{\phi} = p + [q\sin(\phi) + r\cos(\phi)]\tan(\theta),$$

$$\dot{\theta} = q\cos(\phi) - r\sin(\phi),$$

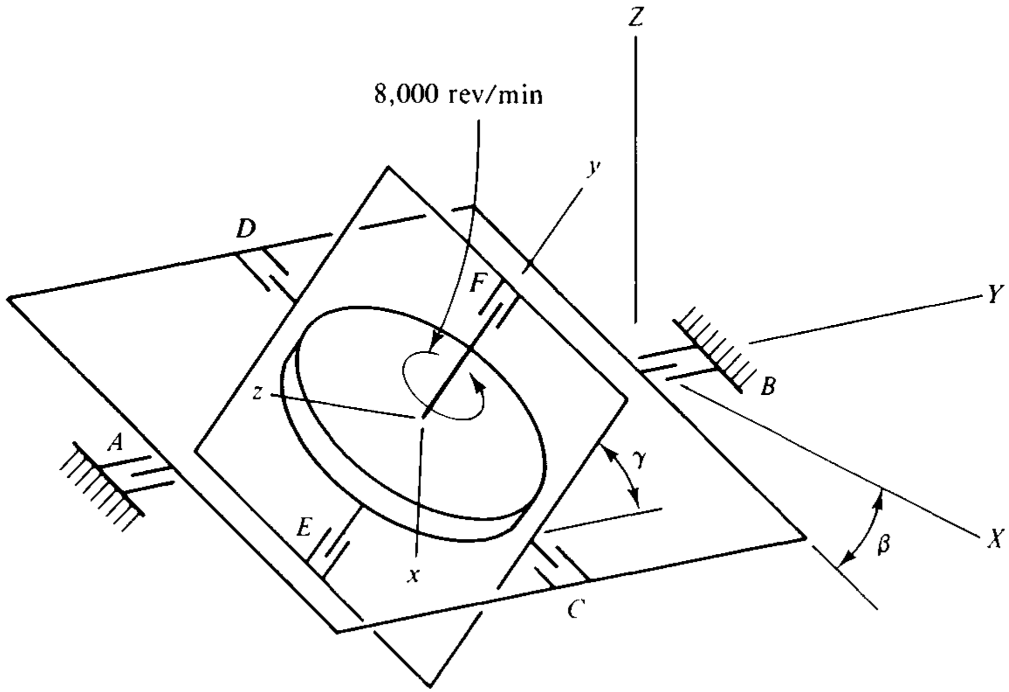
$$\dot{\psi} = [q\sin(\phi) + r\cos(\phi)]\sec(\theta).$$

ZYX or 321

```
function dq_dt=quat(t,q)
wx=10*sin(t);
wy=12*cos(sqrt(2)*t);
wz=11*sin(sqrt(3)*t+pi/4);
q=q/norm(q);

dq_dt=0.5*[wz*q(2)-wy*q(3)+wx*q(4);-
wz*q(1)+wx*q(3)+wy*q(4);wy*q(1)-
wx*q(2)+wz*q(4);-wx*q(1)-wy*q(2)-
wz*q(3)];
```

```
function dangles_dt=Euler(t,angles)
wx=10*sin(t);
wy=12*cos(sqrt(2)*t);
wz=11*sin(sqrt(3)*t+pi/4);

sp=sin(angles(3)); cp=cos(angles(3));
tant=tan(angles(2));  sect=1/cos(angles(2));

dangles_dt=[(wy*sp+wz*cp)*sect;wy*cp-
wz*sp;wx+(wy*sp+wz*cp)*tant];
```

- **Example:** A free gyroscope consists of a flywheel that rotates relative to the inner gimbal at the constant angular speed of 8,000 rev/min, while the rotation of the inner gimbal relative to the outer gimbal is $\gamma = 0.2 \sin(100\pi t)$ rad. The rotation of the outer gimbal is $\beta = 0.5\sin(50\pi t)$ rad. Use the Eulerian angle formulas to determine the angular velocity and angular acceleration of the flywheel at t = 4 ms. Express the results in terms of components relative to the body-fixed xyz and space-fixed XYZ reference frames, where the z axis is parallel to the Z axis at t = 0.



# 4.3 INTERCONNECTIONS

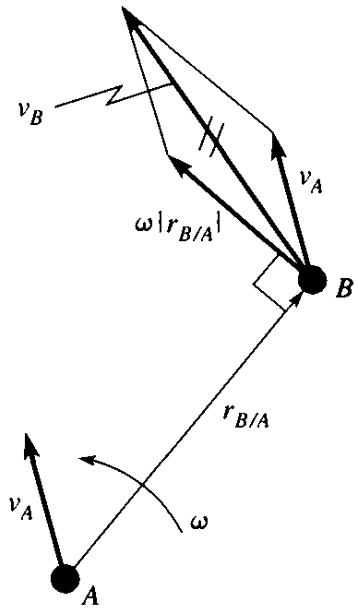
- **Planar motion:** By definition, planar motion means that all points in the body follow parallel planes, which can only happen if the angular velocity is always perpendicular to these planes. Let the X-Y plane of the fixed reference frame and the x-y plane of the body-fixed reference frame be coincident planes of motion.

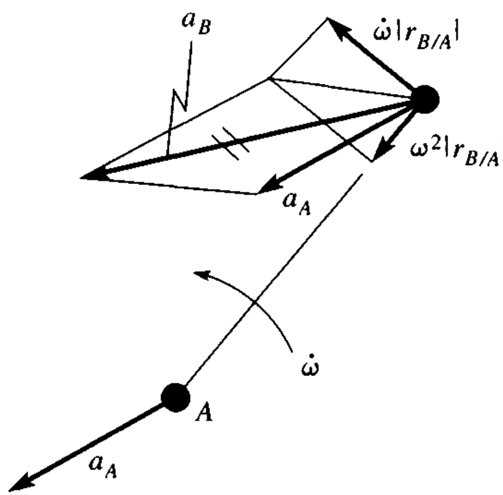
$$\bar{\omega} = \omega \bar{K} = \omega \bar{k}, \qquad \bar{\alpha} = \dot{\omega} \bar{K} = \dot{\omega} \bar{k};$$

$$\bar{r}_{B/A} = X\bar{I} + Y\bar{J} = x\bar{i} + y\bar{j};$$

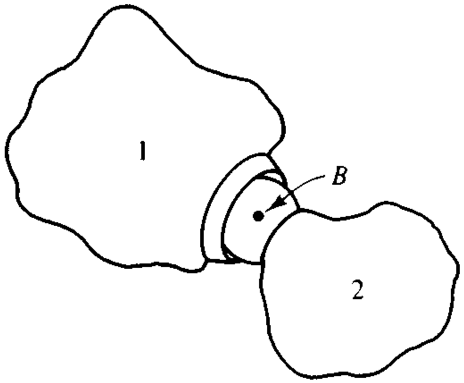
$$\bar{v}_B = \bar{v}_A + \bar{\omega} \times \bar{r}_{B/A},$$

$$\bar{a}_B = \bar{a}_A + \bar{\alpha} \times \bar{r}_{B/A} - \omega^2 \bar{r}_{B/A}.$$



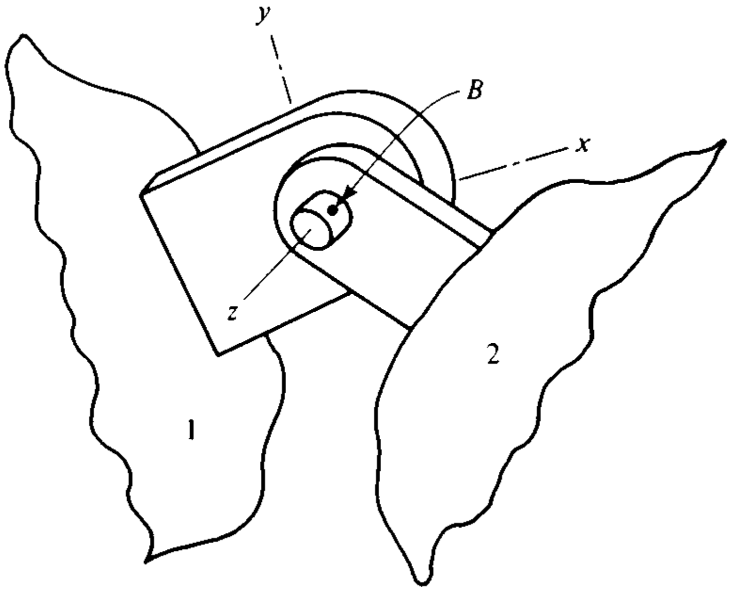



- **Ball-and-socket joint:** it imposes no orientation constraints




$$\bar{v}_{B1} = \bar{v}_{B2}, \qquad \bar{a}_{B1} = \bar{a}_{B2}.$$

- **Pin connection**

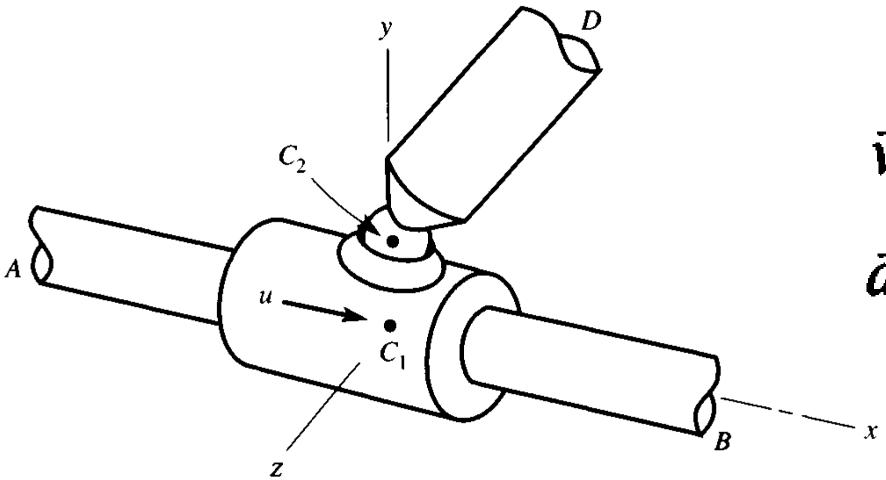



$$\bar{v}_{B1} = \bar{v}_{B2}, \qquad \bar{a}_{B1} = \bar{a}_{B2}.$$

$$\bar{\omega}_2 = \bar{\omega}_1 + \dot{\phi}\bar{k}.$$

$$\bar{\alpha}_2 = \bar{\alpha}_1 + \ddot{\phi}\bar{k} + \dot{\phi}(\bar{\omega}_1 \times \bar{k}).$$

## Collar connection or slider

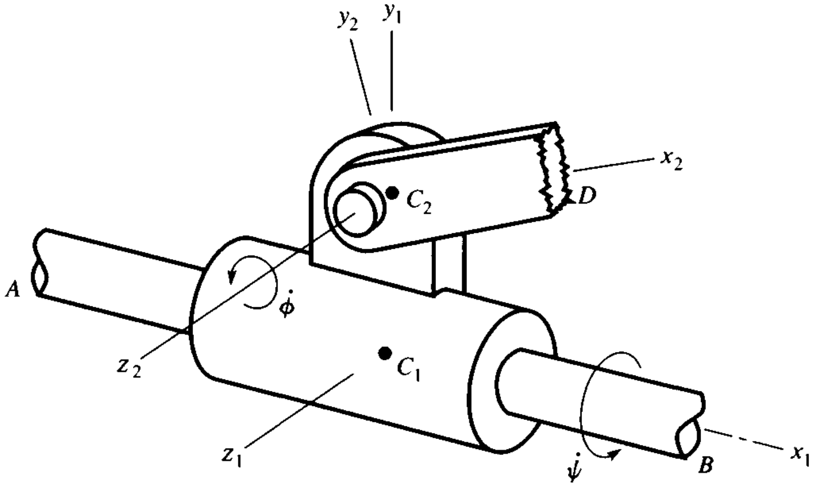



$$(\bar{v}_{C2})_{AB} = u\bar{e}_{B/A}, \qquad (\bar{a}_{C2})_{AB} = \dot{u}\bar{e}_{B/A}.$$

$$\bar{v}_{C2} = \bar{v}_{C1} + u\bar{e}_{B/A},$$

$$\bar{a}_{C2} = \bar{a}_{C1} + \dot{u}\bar{e}_{B/A} + 2\bar{\omega}_{AB} \times u\bar{e}_{B/A}.$$

## Collar with a pin




$$(\bar{v}_{C2})_{AB} = u\bar{e}_{B/A}, \qquad (\bar{a}_{C2})_{AB} = \dot{u}\bar{e}_{B/A}.$$

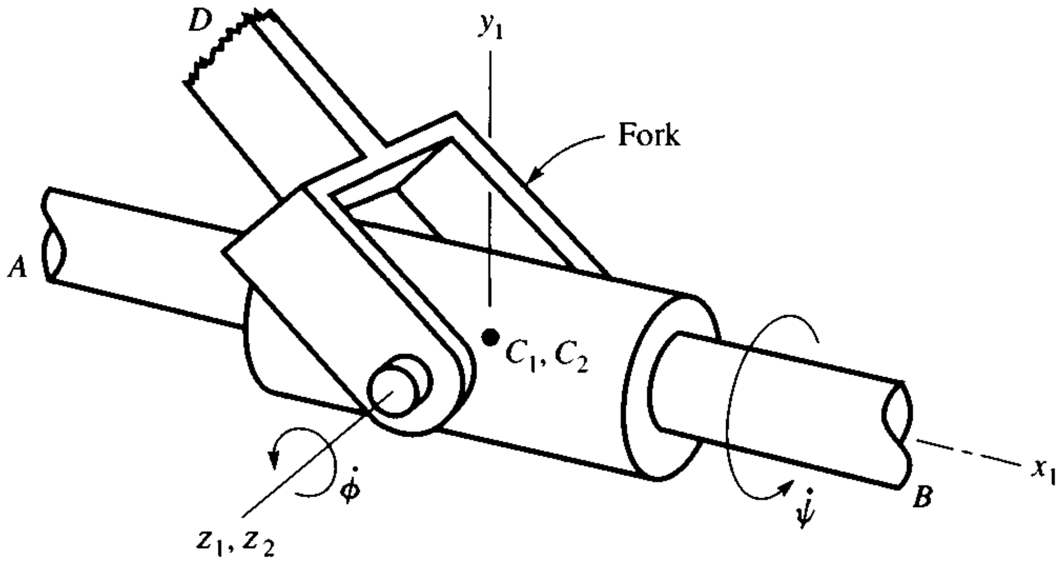
$$\bar{v}_{C2} = \bar{v}_{C1} + u\bar{e}_{B/A},$$

$$\bar{a}_{C2} = \bar{a}_{C1} + \dot{u}\bar{e}_{B/A} + 2\bar{\omega}_{AB} \times u\bar{e}_{B/A}.$$

$$\bar{\omega}_2 = \bar{\omega}_1 + \dot{\psi}\bar{i}_1 + \dot{\phi}\bar{k}_2.$$

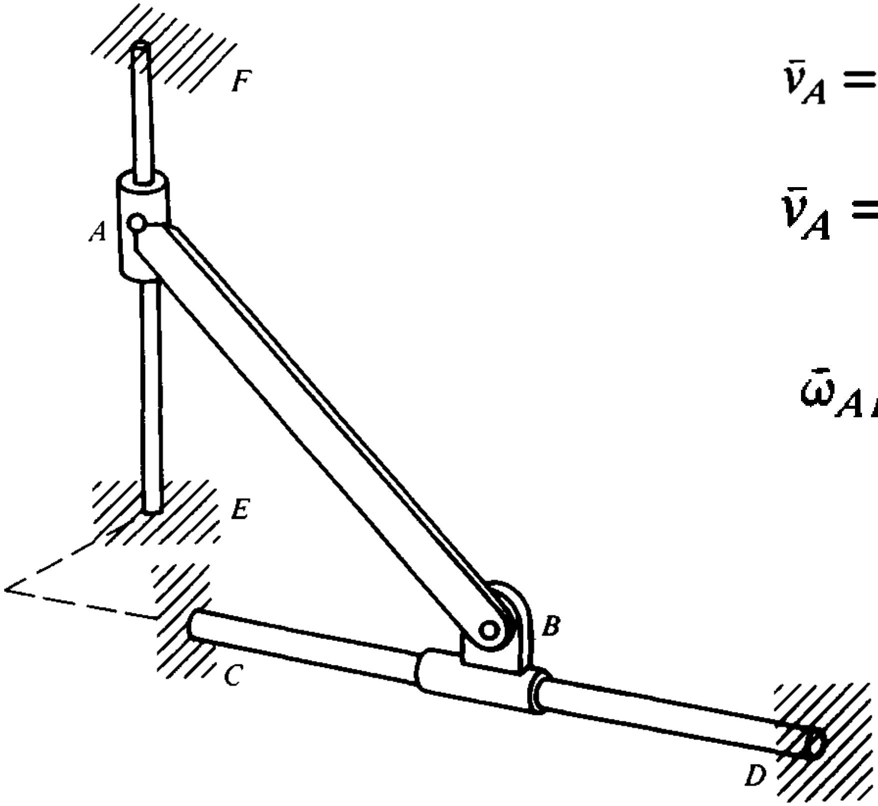
$$\bar{\alpha}_2 = \bar{\alpha}_1 + \ddot{\psi}\bar{i}_1 + \dot{\psi}(\bar{\omega}_1 \times \bar{i}_1) + \ddot{\phi}\bar{k}_2 + \dot{\phi}(\bar{\omega}_2 \times \bar{k}_2).$$

- **Collar with a clevis joint**




**Like the Collar with a pin**

- **Considering a constrained motion**

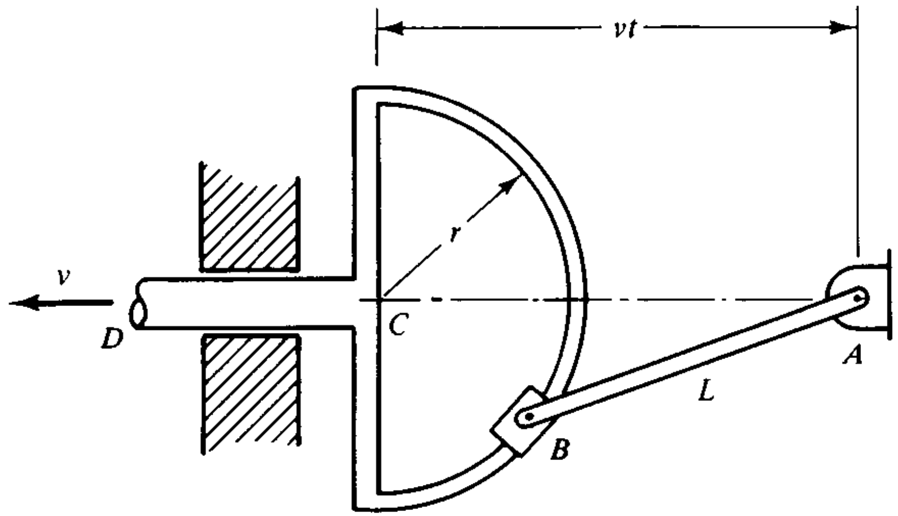



$$\bar{v}_A = u_A \bar{e}_{F/E}, \qquad \bar{v}_B = u_B \bar{e}_{D/C},$$

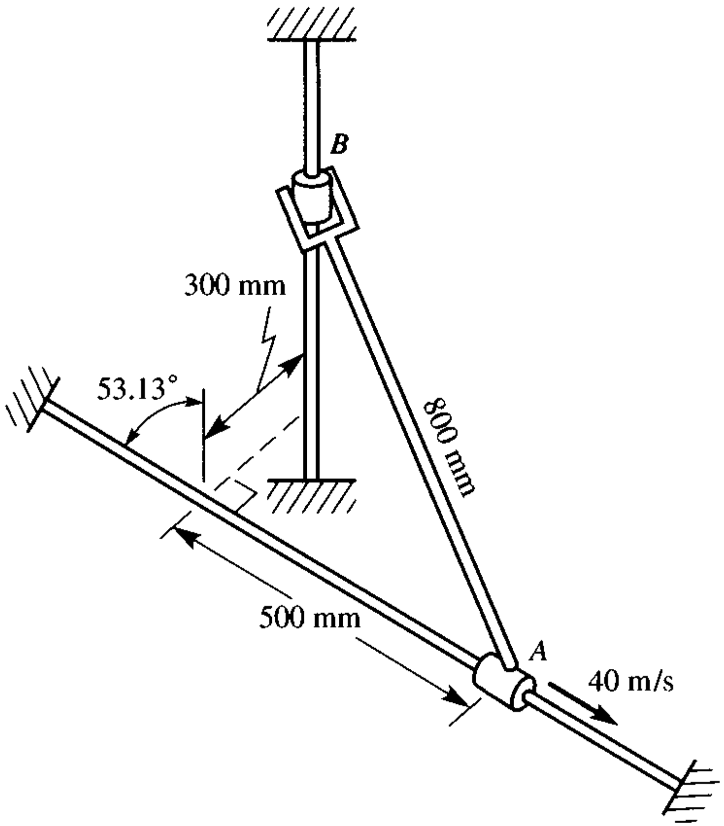
$$\bar{v}_A = \bar{v}_B + \bar{\omega}_{AB} \times \bar{r}_{A/B}.$$

$$\bar{\omega}_{AB} = \dot{\psi} \bar{e}_{D/C} + \dot{\phi} \frac{\bar{r}_{B/A} \times \bar{r}_{D/C}}{|\bar{r}_{B/A} \times \bar{r}_{D/C}|}.$$

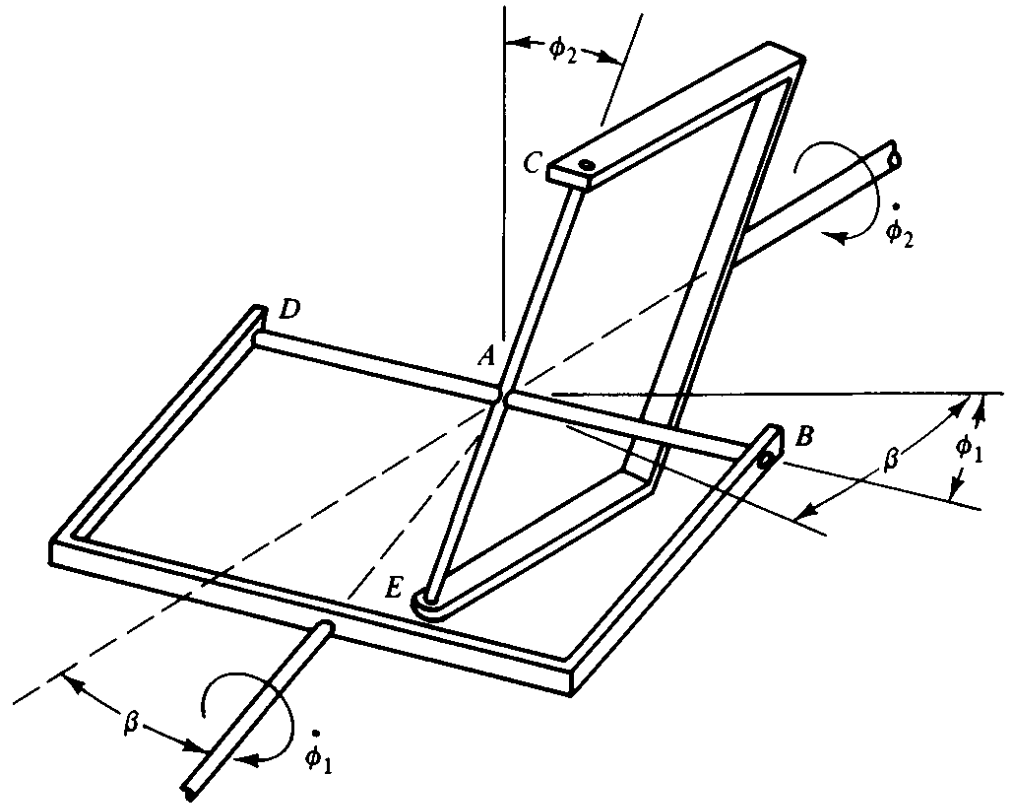
- **Example:** Collar B is pinned to arm AB as it slides over a circular guide bar. The guide bar translates to the left at a constant speed v, such that the distance from pivot A to the center C is vt. Derive expressions for the angular velocity and angular acceleration of arm AB.

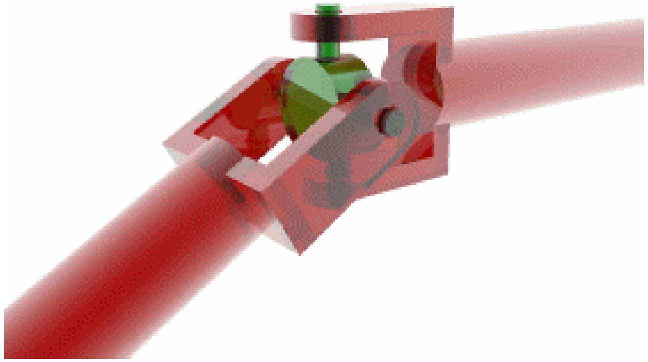


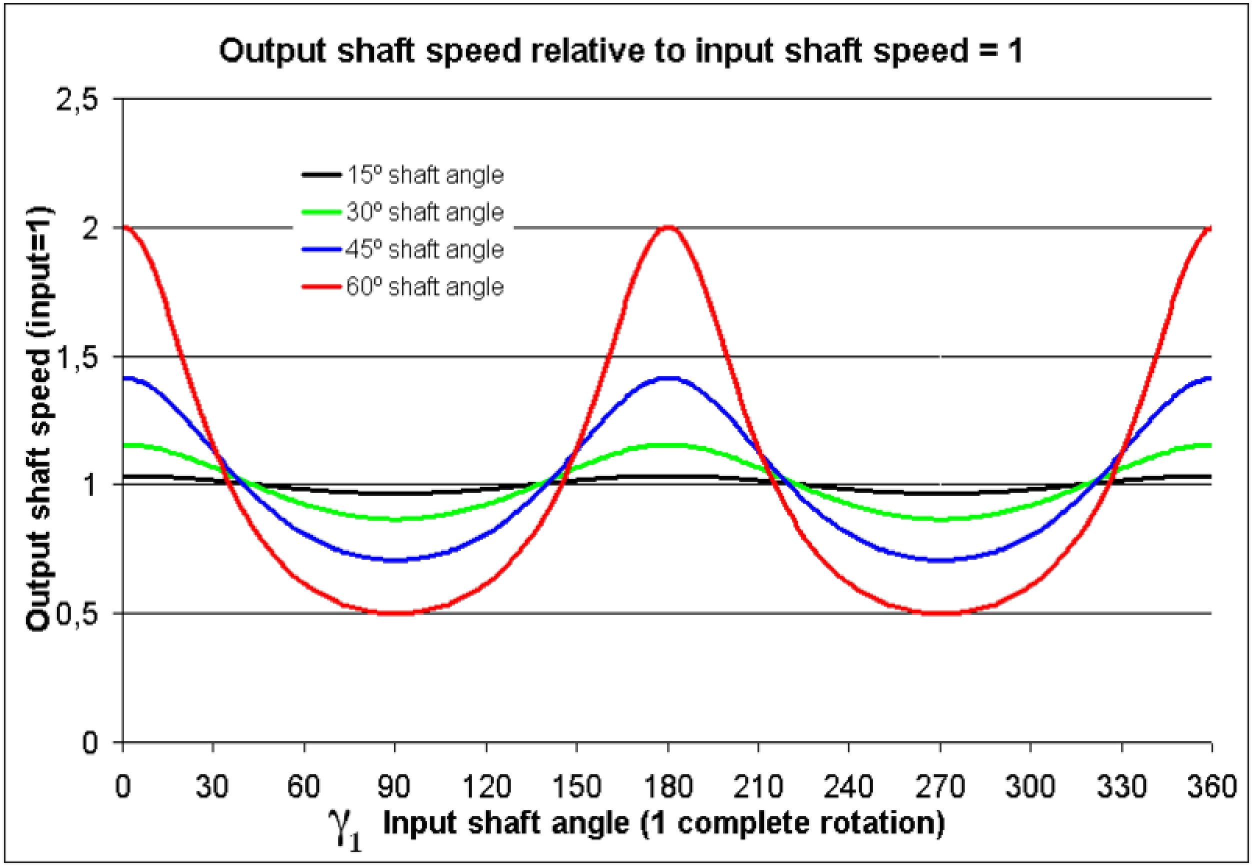
- **Example:** Collar A moves downward and to the right at a constant speed of 40 m/s. The connection of link AB to collar A is a ball-and-socket joint, while that at collar B is a pin. Determine the velocity and acceleration of collar B, and the angular velocity and angular acceleration of bar AB, for the position shown.



- **Example:** Two shafts lying in a common horizontal plane at a skew angle $\beta$ are connected by a cross-link universal joint that is called a cardan joint. Derive an expression for the rotation rate $\omega_2$ in terms of $\omega_1$ and the instantaneous angle of rotation $\phi_1$, where cross-link AB is horizontal when $\phi_1 = 0$.

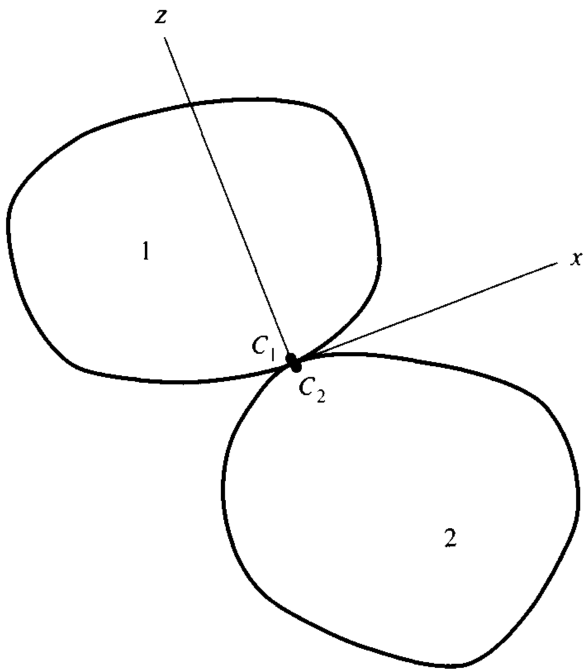





Output shaft speed relative to input shaft speed = 1
Output shaft speed (input=1)
2,5
2
1,5
1
0,5
0
15º shaft angle
30º shaft angle
45º shaft angle
60º shaft angle
0
30
60
90
120
150
180
210
240
270
300
330
360
$\gamma_1$ Input shaft angle (1 complete rotation)

# 4.4 ROLLING

- A common constraint condition arises when bodies rotate as they move over each other. The fact that the contacting surfaces cannot penetrate each other imposes a restriction on the velocity components perpendicular to the plane of contact (that is, the tangent plane).

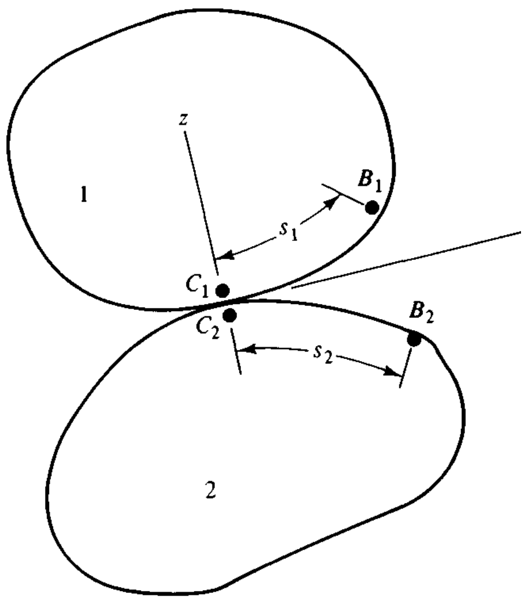



- Because the surface of each body is impenetrable, the velocity components normal to the contact plane must match. Let $C_1$ and $C_2$ be contacting points on each body. Then $\bar{v}_{C1} \cdot \bar{k} = \bar{v}_{C2} \cdot \bar{k}.$

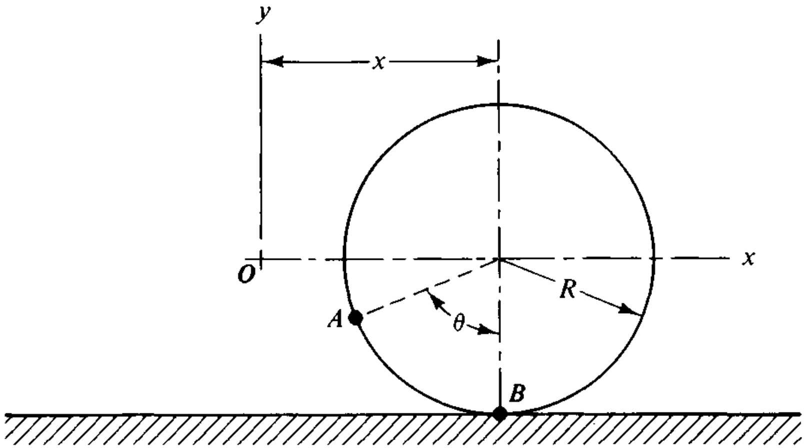
- The special case of rolling without slipping imposes an additional constraint.

$$\bar{v}_{C1} = \bar{v}_{C2} \quad \text{for no slipping.}$$

- The condition of no slipping impose a constraint that the arclength $s_1$ along the perimeter of body 1 between the points $B_1$ and $C_1$ is the same as the arclength $s_2$ along body 2 between points $B_2$ and $C_2$.

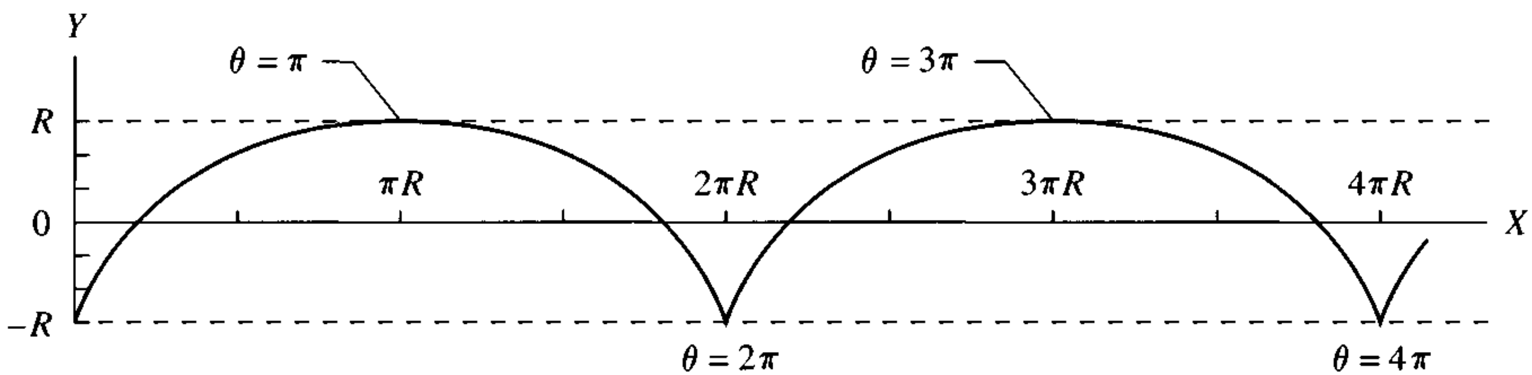


- The most common application of such a description is for a wheel rolling along the ground.




$$x = R\theta.$$

$$\bar{r}_{A/O} = R(\theta - \sin\theta)\bar{i} - (R\cos\theta)\bar{j}.$$

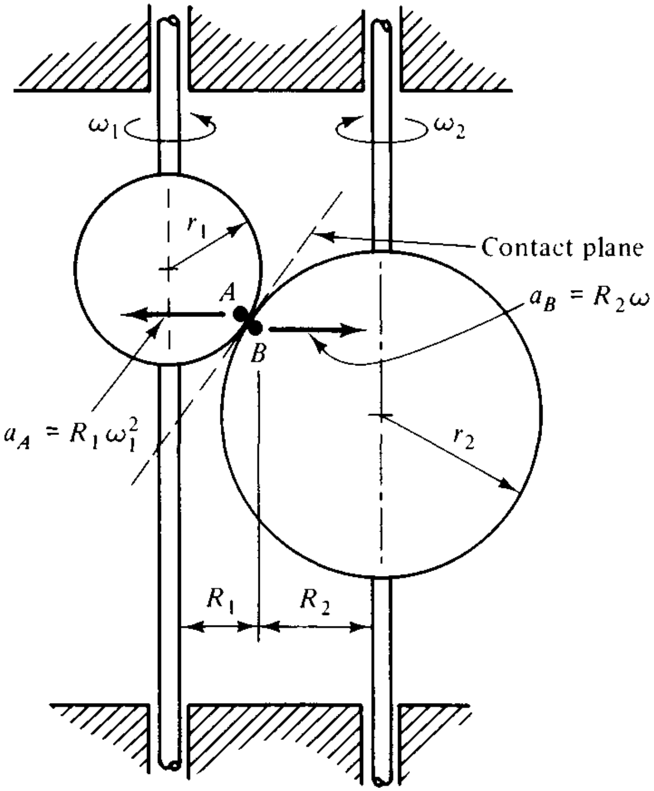



$$\bar{v}_A = v(1-\cos\theta)\bar{i} + v(\sin\theta)\bar{j},$$

$$\bar{a}_A = \left[\dot{v}(1-\cos\theta) + \frac{v^2}{R}\sin\theta\right]\bar{i} + \left[\dot{v}\sin\theta + \frac{v^2}{R}\cos\theta\right]\bar{j}.$$

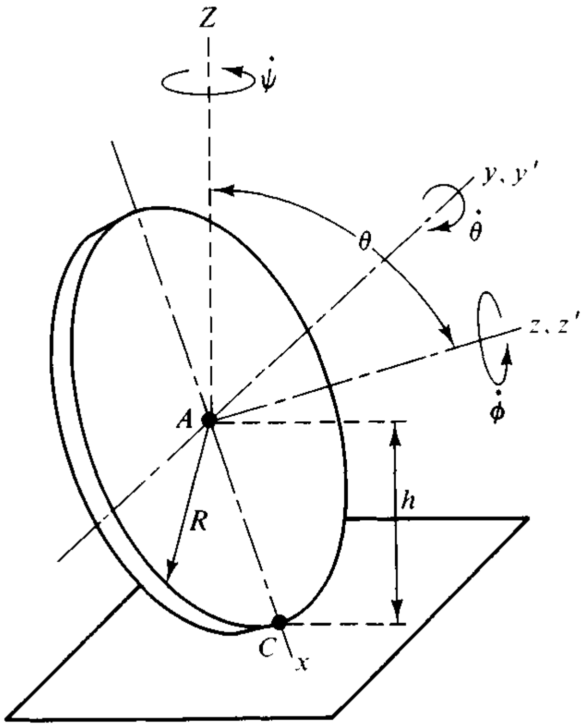
## ❖ What about acceleration?

- Acceleration is more complicated, because the contacting points on each body come together and then separate.
- A common misconception arises from the case of the rolling wheel on the surface.

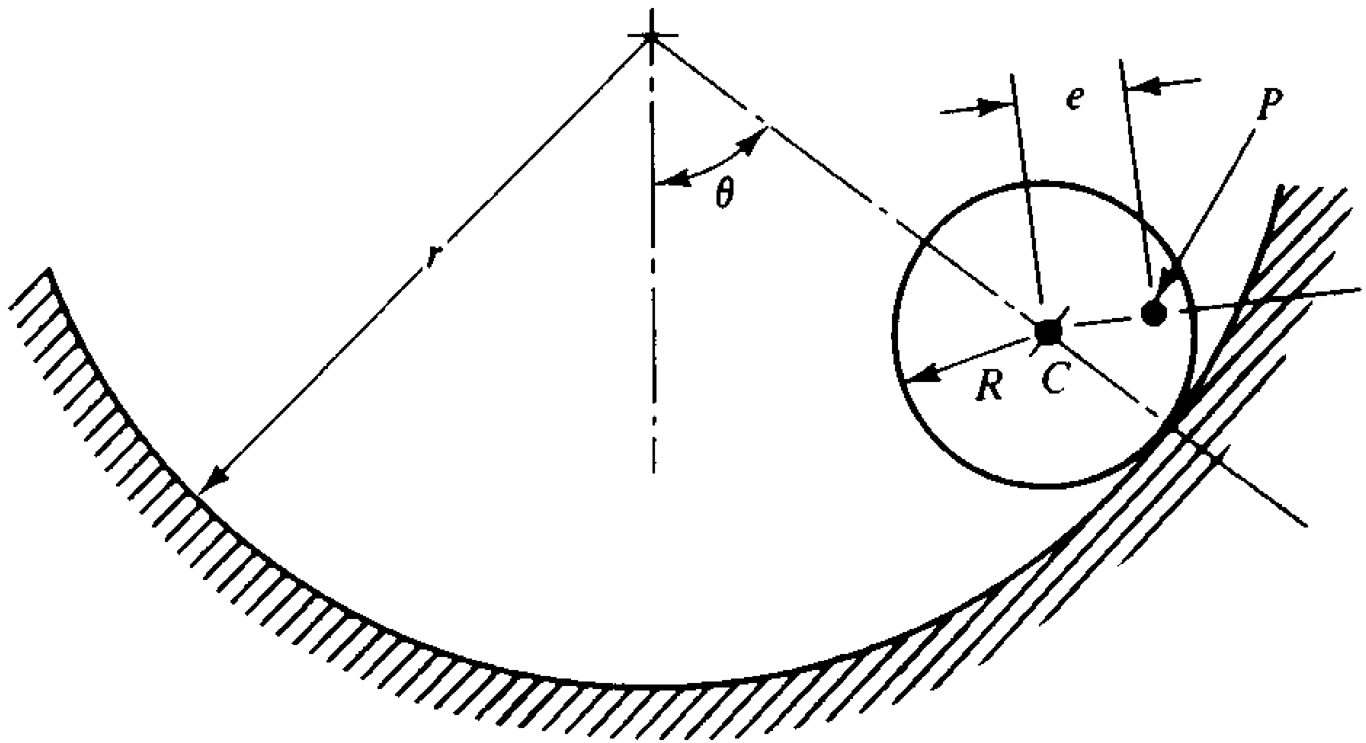


- **Example: Rolling a coin on surface**.

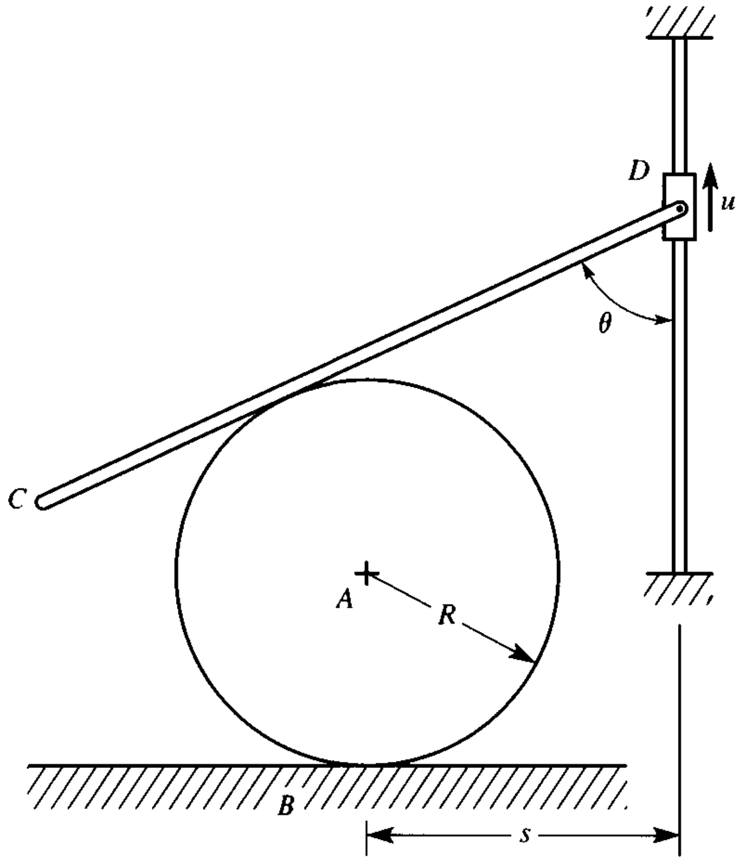
Derive the velocity and acceleration for the center of coin as a function of euler angles and their rates.



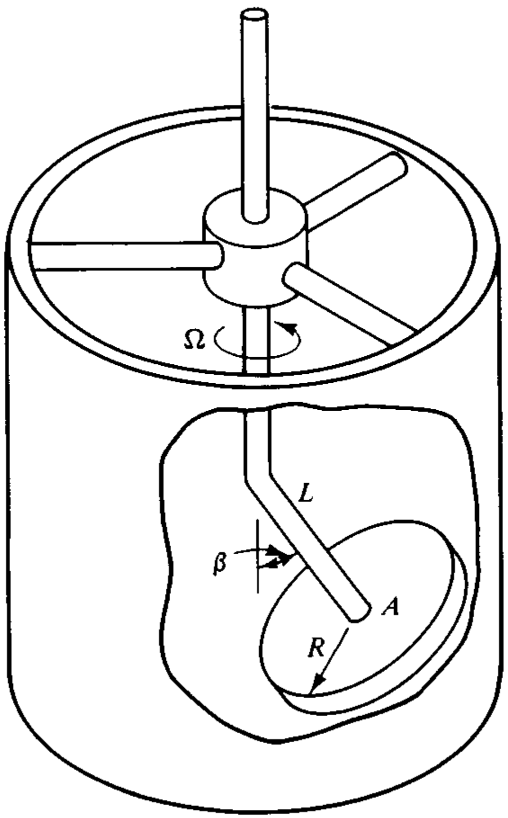
- **Example:** The cylinder of radius R rolls without slipping inside a semi cylindrical cavity. Point P is collinear with the vertical centerline when the vertical angle $\theta$ locating the cylinder's center C is zero. Derive expressions for the velocity and acceleration of point P in terms of $\phi$ and the speed $v$ of the center C.

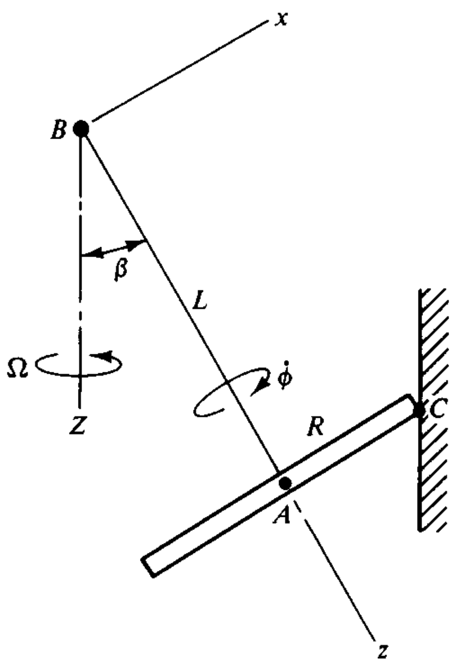


- **Example:** Rack CD, which meshes with gear A, is actuated by moving collar D upward at the constant speed u. Rack B, over which gear A rolls, is stationary. Derive expressions for the velocity and acceleration of the center of gear A in terms of the angle $\theta$ and the distance s.



- **Example:** The shaft of disk A rotates about the vertical axis at the constant rate Ω as the disk rolls without slipping over the inner surface of the cylinder. Determine the angular velocity and angular acceleration of the disk and the acceleration of the point on the disk that contacts the cylinder.



- **Example:** A disk rolls without slipping on the X-Y plane. At the instant shown, the horizontal diameter ACB is parallel to the X axis. Also at this instant, the horizontal components of the velocity of the center C are known to be 5 m/s in the X direction and 3 m/s in the Y direction, while the Y component of the velocity of point B is 6 m/s. Determine the precession, nutation, and spin rates for the Eulerian angles.