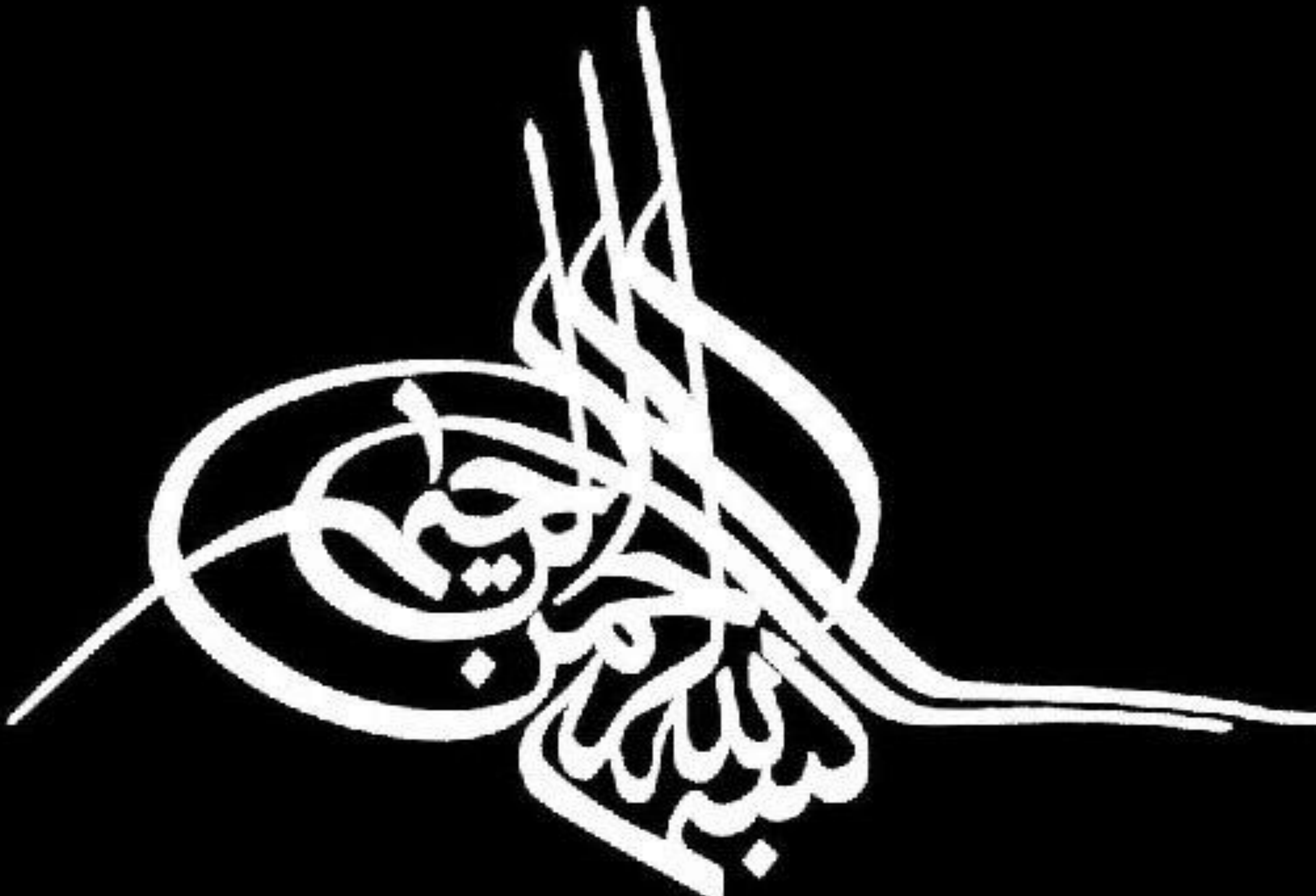

بسم الله الرحمن الرحيم

NEWTONIAN MECHANICS
PARTICLES KINETICS

Purpose:

- **To Study Kinetic States and Principles of Particles .**

Topics:

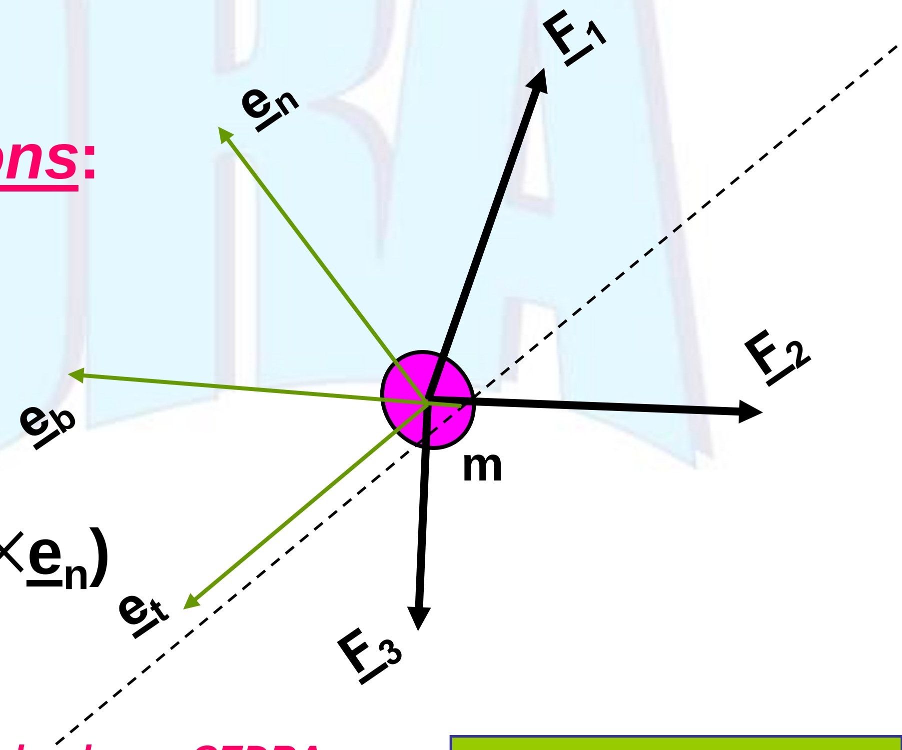
- **Kinetic States:**
  - **Momentum (*Linear Momentum*).**
  - **Moment of Momentum (*Angular Momentum*).**
- **Kinetic Principles:**
  - **Momentum Principle.**
  - **Moment of Momentum Principle.**
- **Differential Equations of Motion.**

Kinetics: Study and analysis of forces causing the motion.

We studied the Newtonian Laws in Chapter Two. Let us again consider the *Law of Motion*:

$$\sum \underline{F} = m\underline{a} \qquad (6.1)$$

Equations of Rectilinear Motions:



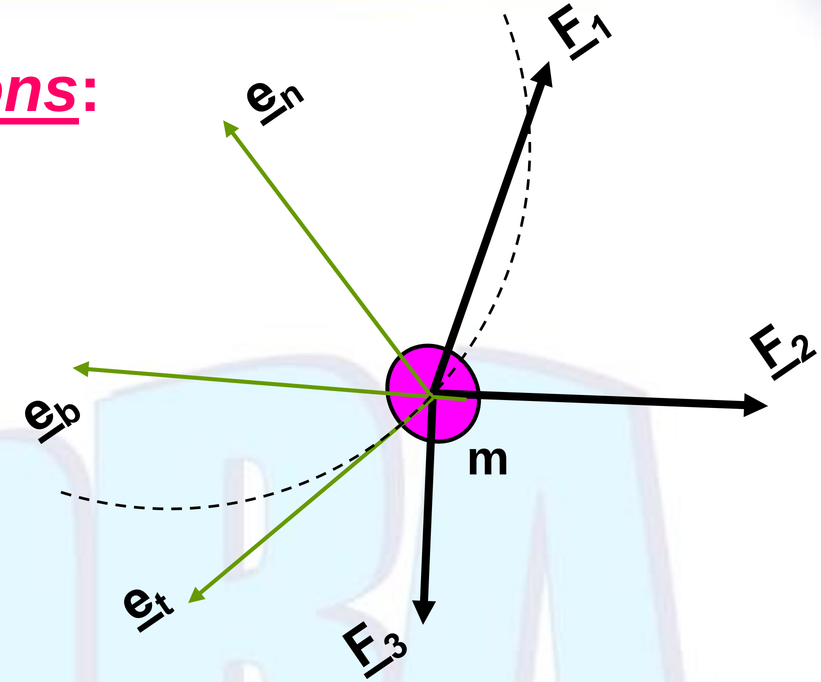
$\underline{e}_t$: Tangential Unit Vector,
$\underline{e}_n$: Normal Unit Vector, and
$\underline{e}_b$: Binormal Unit Vector = ($\underline{e}_t \times \underline{e}_n$)

$$\sum \underline{F} = (\sum F_t)\underline{e}_t + (\sum F_n)\underline{e}_n + (\sum F_b)\underline{e}_b = m\underline{a} = m(\dot{v}\underline{e}_t)$$

$$\begin{Bmatrix} \sum F_t = m\dot{v} \\ \sum F_n = 0 \\ \sum F_b = 0 \end{Bmatrix} \qquad \textbf{(6.2)}$$

Note: In rectilinear motion, forces in "$\underline{e}_n$" and "$\underline{e}_b$" directions are balanced.

Equations of Curvilinear Motions:



Path Variable Concept:

$$\underline{v} = \dot{s}\underline{e}_t = v\underline{e}_t$$

$$\underline{a} = \dot{v}\underline{e}_t + \frac{v^2}{\rho}\underline{e}_n$$

$$\sum \underline{F} = (\sum F_t)\underline{e}_t + (\sum F_n)\underline{e}_n + (\sum F_b)\underline{e}_b = m\underline{a} = m(\dot{v}\underline{e}_t + \frac{v^2}{\rho}\underline{e}_n)$$

$$\begin{cases} \sum F_t = m\dot{v} \\ \sum F_n = m\dfrac{v^2}{\rho} \\ \sum F_b = 0 \end{cases} \qquad (6.3)$$

Rectangular Cartesian Coordinates:

$$\underline{a} = \ddot{x}\underline{i} + \ddot{y}\underline{j} + \ddot{z}\underline{k} = a_x\underline{i} + a_y\underline{j} + a_z\underline{k}$$

$$\begin{cases} \sum F_x = m\ddot{x} = ma_x \\ \sum F_y = m\ddot{y} = ma_y \\ \sum F_z = m\ddot{z} = ma_z \end{cases} \qquad (6.4)$$

Cylindrical Coordinates:

$$\underline{a} = a_R \underline{e}_R + a_\varphi \underline{e}_\varphi + a_z \underline{e}_z = (\ddot{R} - R\dot{\varphi}^2)\underline{e}_R + (R\ddot{\varphi} + 2\dot{R}\dot{\varphi})\underline{e}_\varphi + \ddot{z}\underline{k}$$

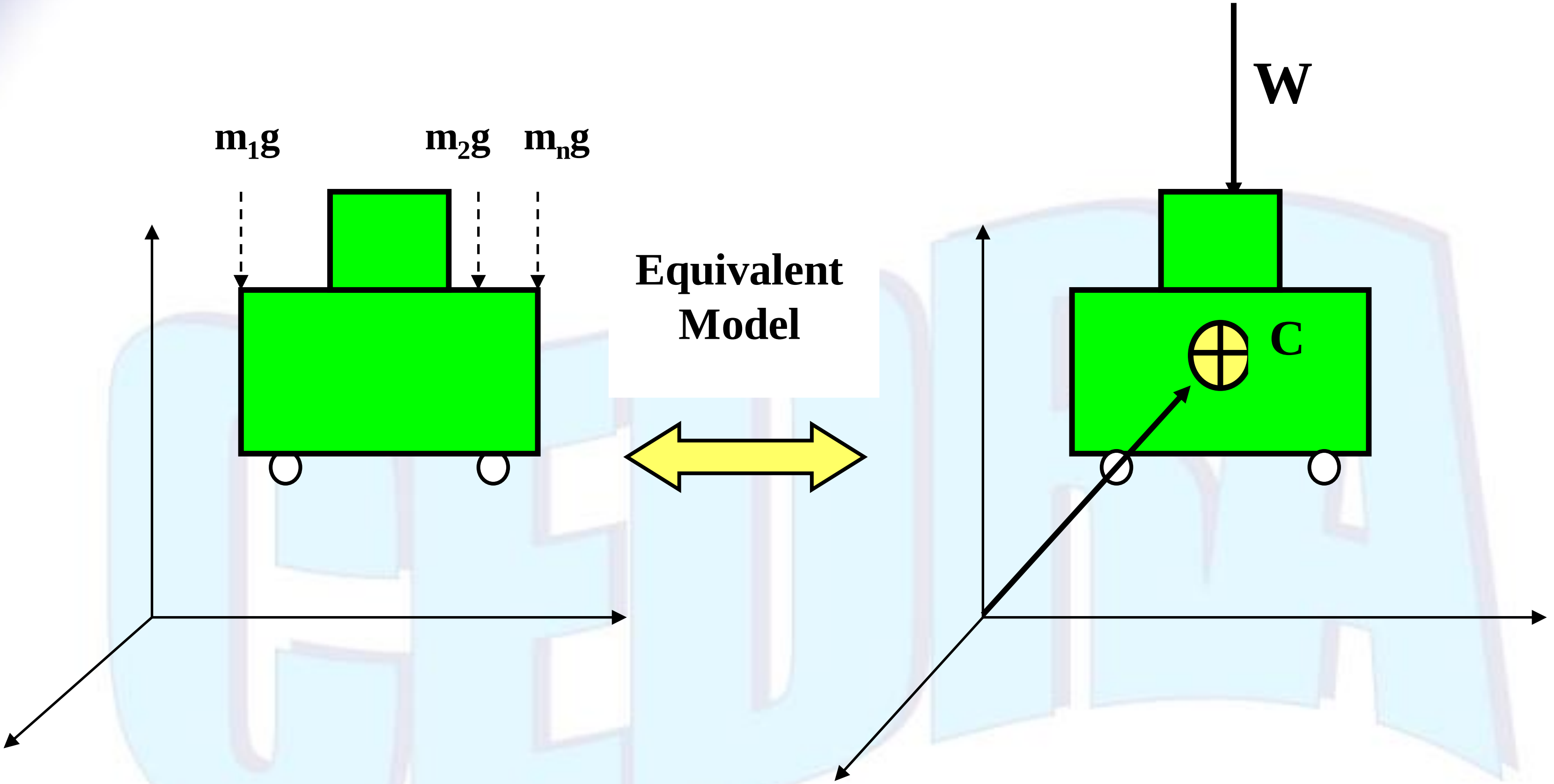
$$\begin{cases} \sum F_R = ma_R = m(\ddot{R} - R\dot{\varphi}^2) \\ \sum F_\varphi = ma_\varphi = m(R\ddot{\varphi} + 2\dot{R}\dot{\varphi}) \\ \sum F_z = ma_z = m\ddot{z} \end{cases} \quad (6.5)$$

Spherical Coordinates:

$$\underline{a} = a_R \underline{e}_R + a_\varphi \underline{e}_\varphi + a_\theta \underline{e}_\theta$$

$$\begin{cases} \sum F_R = ma_R \\ \sum F_\varphi = ma_\varphi \\ \sum F_\theta = ma_\theta \end{cases} \quad (6.6)$$

Newton's Equation of Motion for a System of Particles:

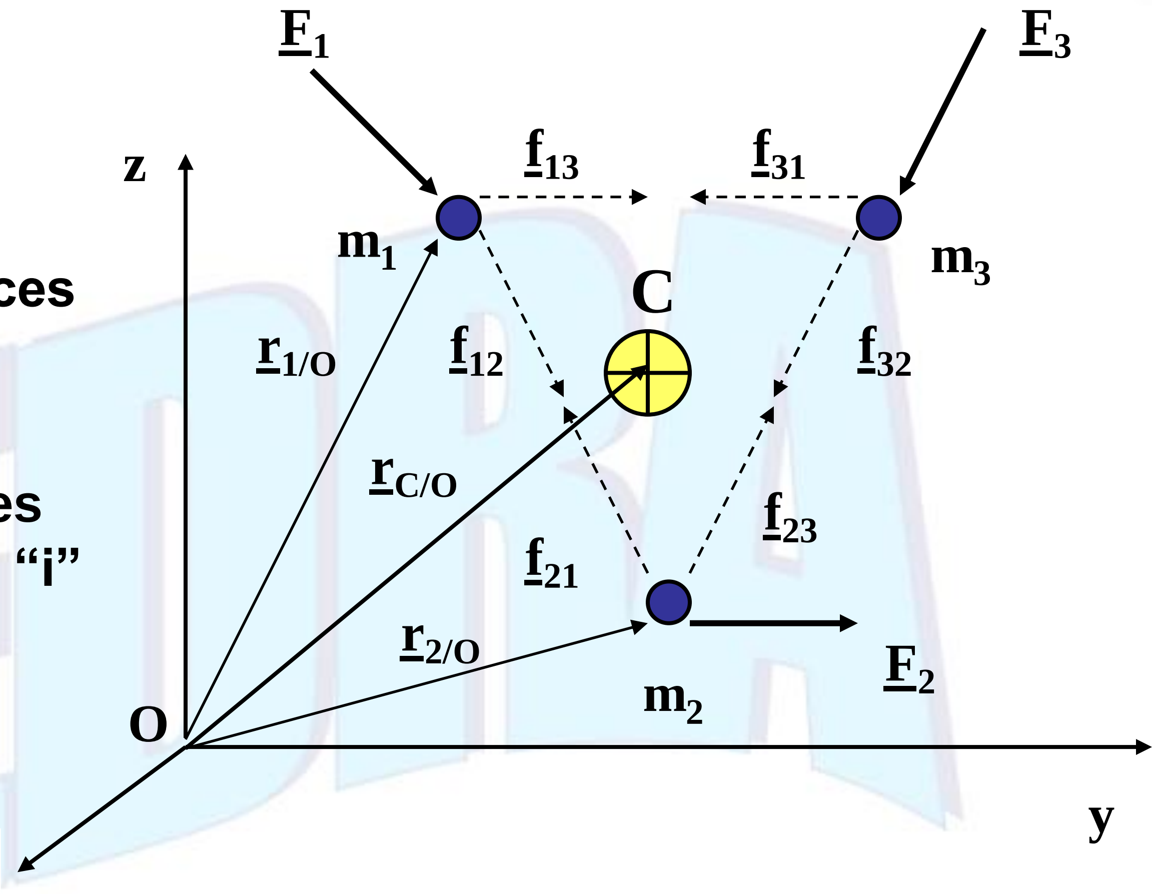


Total Weight = $W = mg$ **, where:** $m = \sum_{i=1}^{n} m_i$ **(6.7)**

Consider a system of particles, where:

$\underline{F}_i$: Resultant force on particle "i" from sources external to system.

$\underline{f}_{ij}\,(i \neq j)$: Interaction forces on each particle "i" due to "j".



Applying Newton's 2nd law, we have: $\sum \underline{F} = m\underline{a}$

$$\sum_{i=1}^{n} \underline{F}_i + \sum_{\substack{i=1 \\ i \neq j}}^{n} \underline{f}_{ij} = \sum_{i=1}^{n} m_i \ddot{\underline{r}}_{i/O} \quad \Rightarrow \quad \sum_{i=1}^{n} \underline{F}_i = \sum_{i=1}^{n} m_i \ddot{\underline{r}}_{i/O} = \frac{d^2}{dt^2} \sum_{i=1}^{n} m_i \underline{r}_{i/O}$$

0

$$because(\underline{f}_{ij} = -\underline{f}_{ji})$$

but the location of the center of mass is found from:

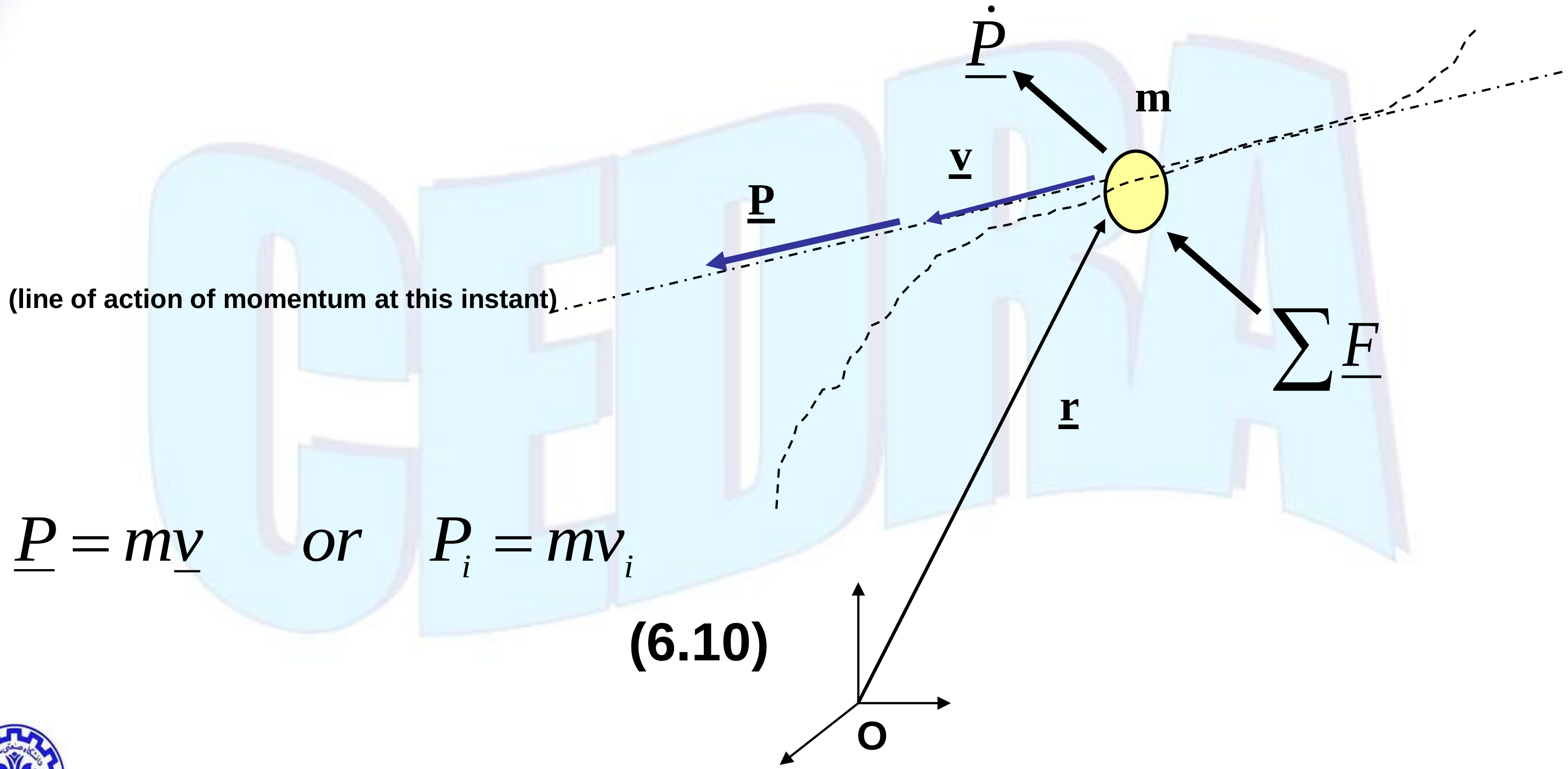
$$m\underline{r}_{C/O} = \sum_{i=1}^{n} m_i \underline{r}_{i/O} \qquad \textbf{(6.8)}$$

$$\underline{F}_{total} = \sum_{i=1}^{n} \underline{F}_i = \frac{d^2}{dt^2} m\underline{r}_{C/O} = m\ddot{\underline{r}}_{C/O} = m\underline{a}_C \qquad \textbf{(6.9)}$$

❖ ***When we model a system of particles as a single particle, we are actually studying the motion of its center of mass.***

The Kinetic States:

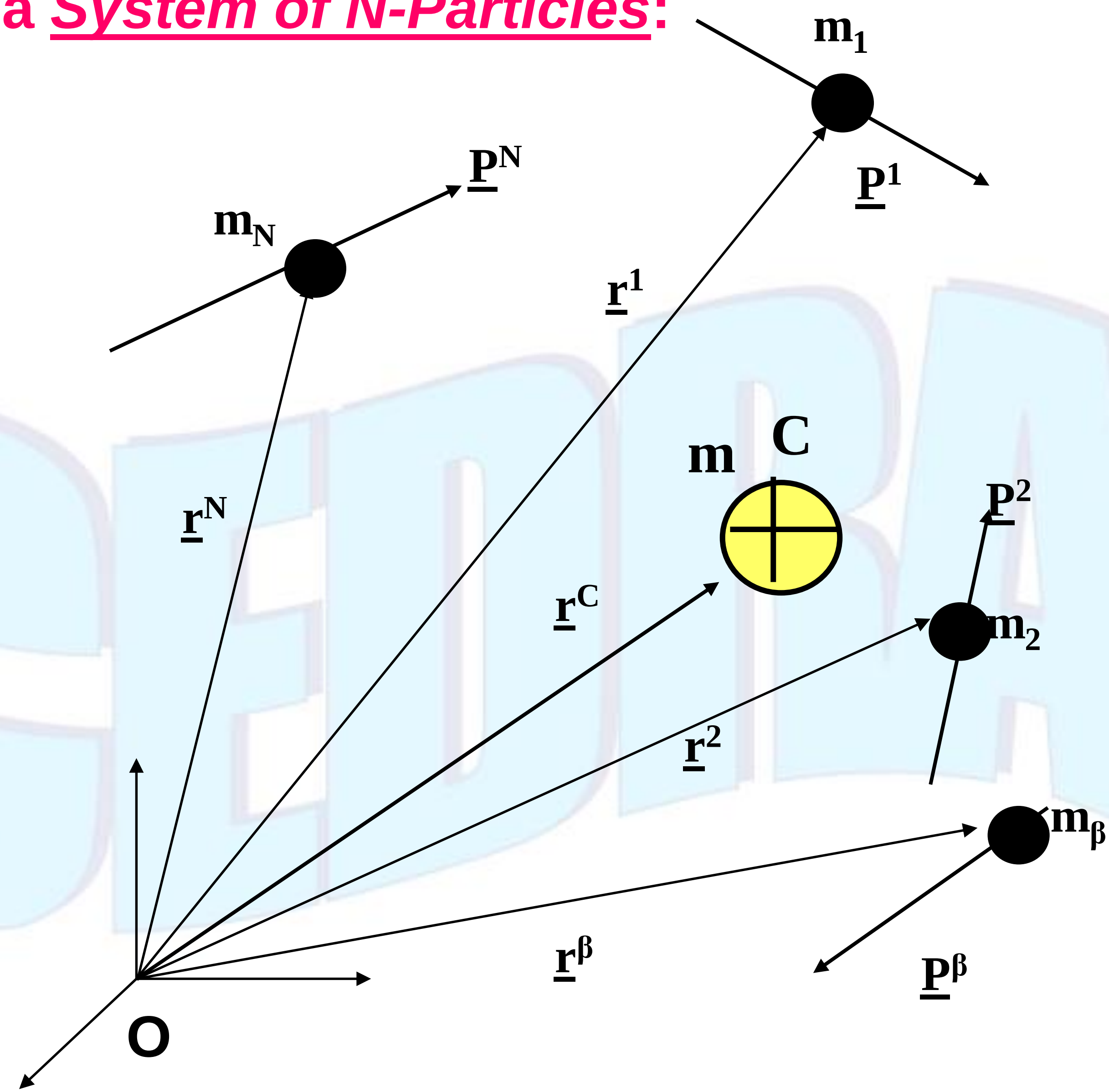
- ***Linear Momentum (Momentum)***: **For a Single Particle is defined by the mass times the velocity of the particle.**



$$\underline{P} = m\underline{v} \quad or \quad P_i = mv_i$$

(6.10)

Consider a ***System of N-Particles***:



- ***For a System of N-Particles***:

$$\underline{P} = \sum_{\beta=1}^{N} \underline{P}^{\beta} = \sum_{\beta=1}^{N} m_{\beta} \underline{v}^{\beta}, \quad or$$

(6.11)

$$P_i = \sum_{\beta=1}^{N} P_i^{\beta} \qquad (i = 1,2,3)$$

- ***Moment of Momentum (Angular Momentum)***: **For a Single Particle**;

$$\underline{H}^{O} = \underline{r} \times \underline{P}, \quad or$$

$H_i^{O} = \gamma_{ijk} x_j P_k$, **where: "O" is a moment center. (6.12)**

- ***For a System of N-Particles***:

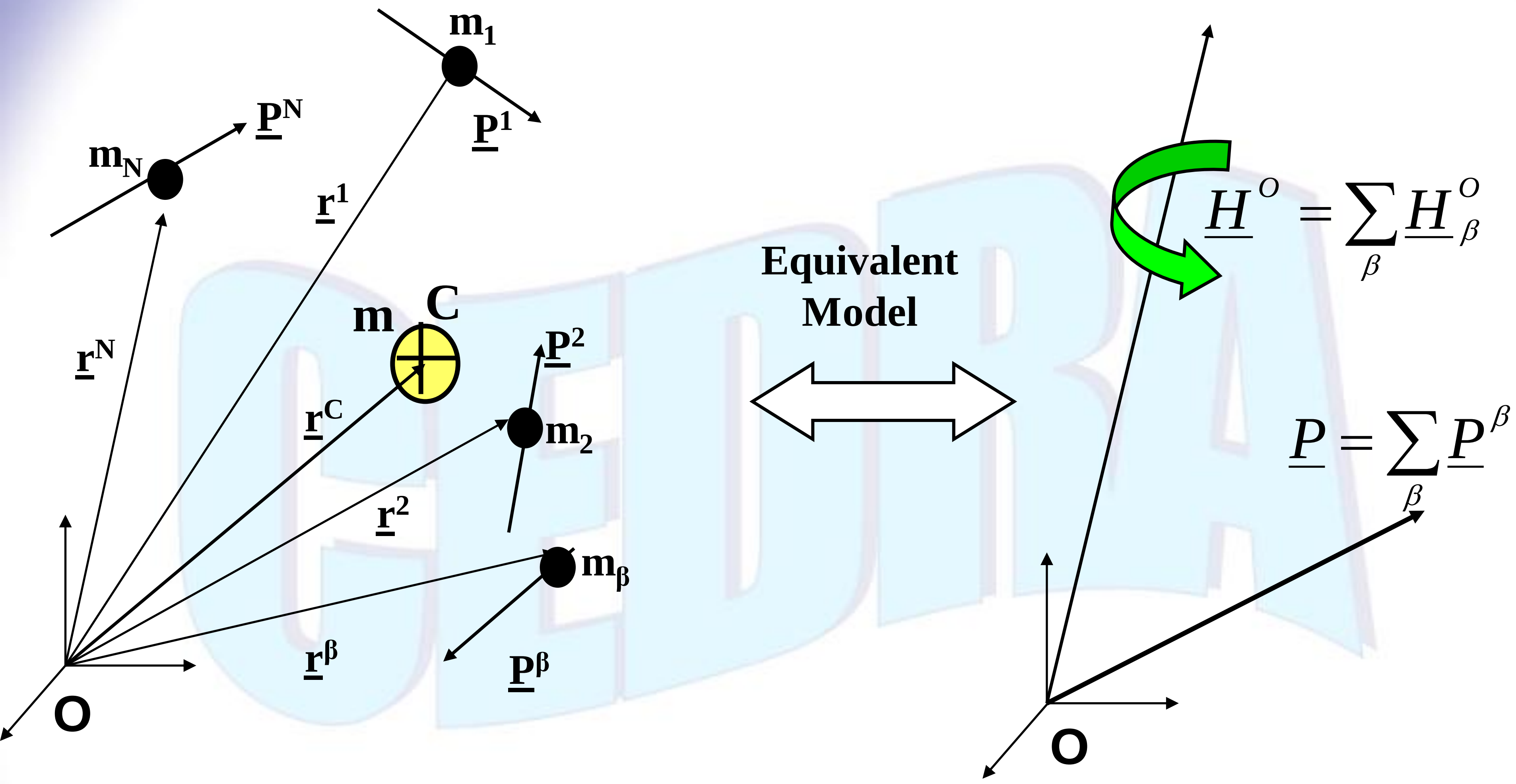
$$\underline{H}^{O} = \sum_{\beta=1}^{N} \underline{H}_{\beta}^{O} = \sum_{\beta=1}^{N} \underline{r}^{\beta} \times \underline{P}^{\beta} = \sum_{\beta=1}^{N} m_{\beta} \underline{r}^{\beta} \times \underline{v}^{\beta}, \quad or$$

$$H_{i}^{\beta} = \sum_{\beta=1}^{N} \gamma_{ijk} x_{j}^{\beta} P_{k}^{\beta} \qquad (i, j, k \equiv 1,2,3) \qquad \textbf{(6.13)}$$

where:

$\underline{r}^{\beta}$ **: position vector of the "βth" particle from the moment center "O".**

The *equivalent system* may be represented as follows:



Where:

$$m = \sum_{\beta=1}^{N} m_\beta , \quad and$$

(6.14)

$$m\underline{r}^{C} = \sum_{\beta=1}^{N} m_\beta \underline{r}^{\beta} \quad or \quad m x_i^{C} = \sum_{\beta=1}^{N} m_\beta x_i^{\beta}$$

- ***For a Continuum (i.e. Rigid Body)*:**

$$m = \int_m dm , \quad and$$

(6.15)

$$x_i^{C} = \frac{1}{m} \int_m x_i dm$$

and,

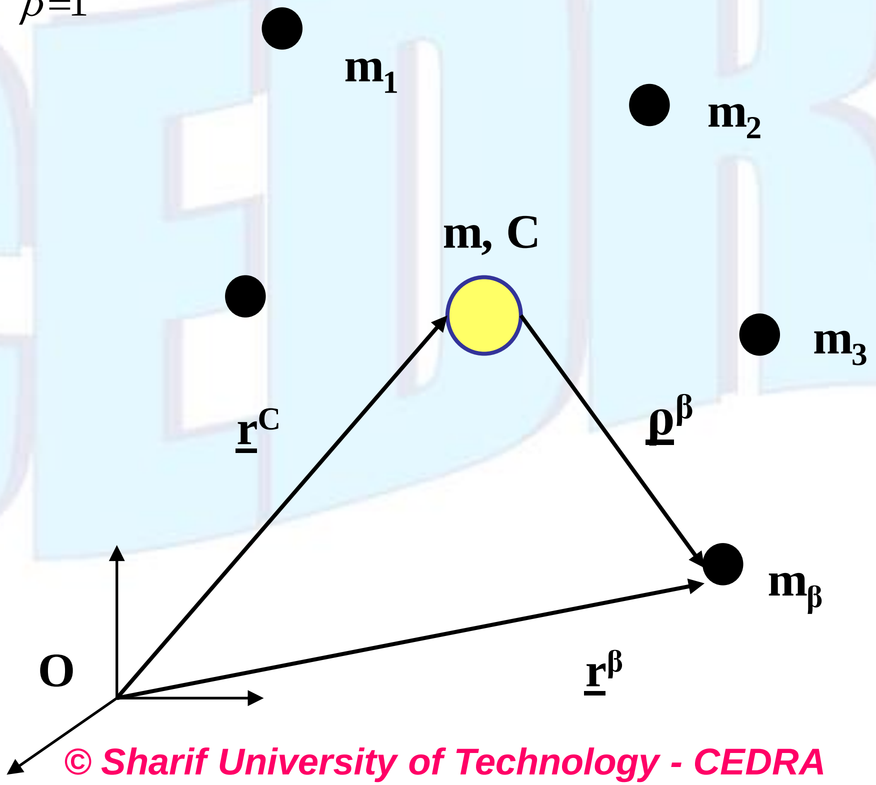
$$\underline{P} = \int_m \underline{v} dm , \quad and$$

(6.16)

$$\underline{H}^{O} = \int_m \underline{r} \times \underline{v} dm$$

***Theorem-16*:** **If the individual masses in a system of particles are constant, then the *momentum of the equivalent mass particle* is equal to the *Total/Global Momentum* of the system of particles.**

$$\underline{P} = m\underline{v}^{C} = \sum_{\beta=1}^{N} m_{\beta}\underline{v}^{\beta} = \text{(Global Momentum)} \qquad (6.17)$$



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Proof:

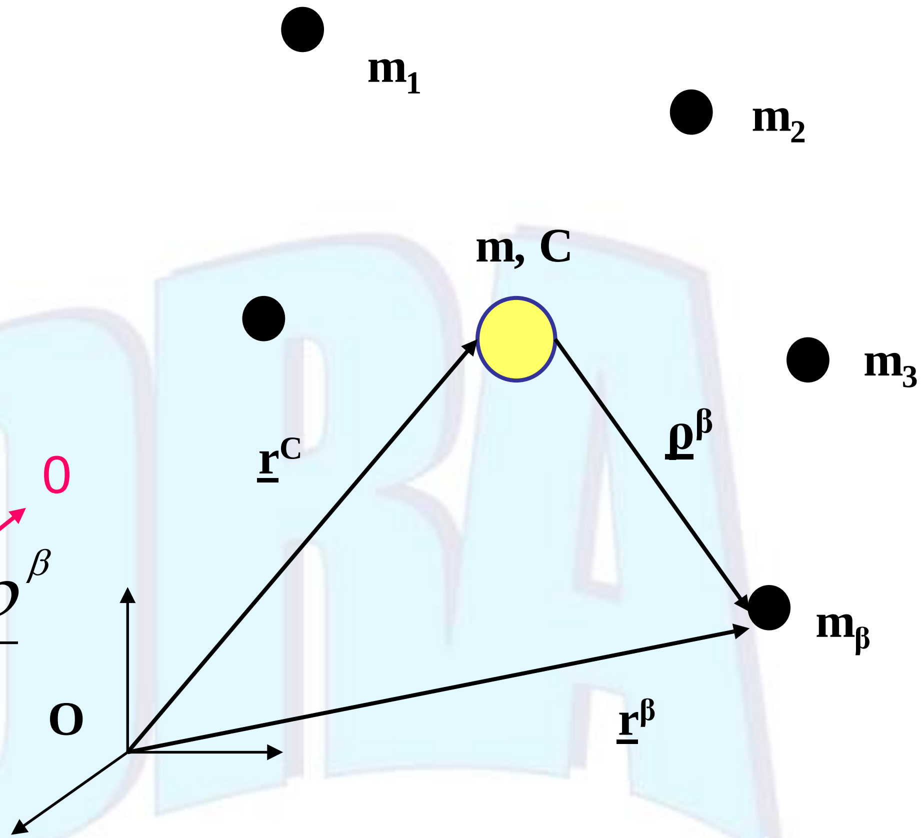
By definition, we have:

$$\underline{P} = \frac{d}{dt}\sum_{\beta=1}^{N} m_\beta \underline{r}^\beta =$$

$$= \frac{d}{dt}\sum_{\beta=1}^{N} m_\beta (\underline{r}^C + \underline{\rho}^\beta)$$

$$= \frac{d}{dt}\sum_{\beta=1}^{N} m_\beta \underline{r}^C + \frac{d}{dt}\sum_{\beta=1}^{N} m_\beta \underline{\rho}^\beta \quad (\to 0)$$

$$= m\underline{v}^C + 0 = \sum_{\beta=1}^{N} m_\beta \underline{v}^\beta$$



Note: the first moment of a mass system about its mass center vanishes, meaning:

$$\sum_{\beta=1}^{N} m_\beta \underline{\rho}^\beta = 0$$

Global Moment of Momentum, and the Central Moment of Momentum:

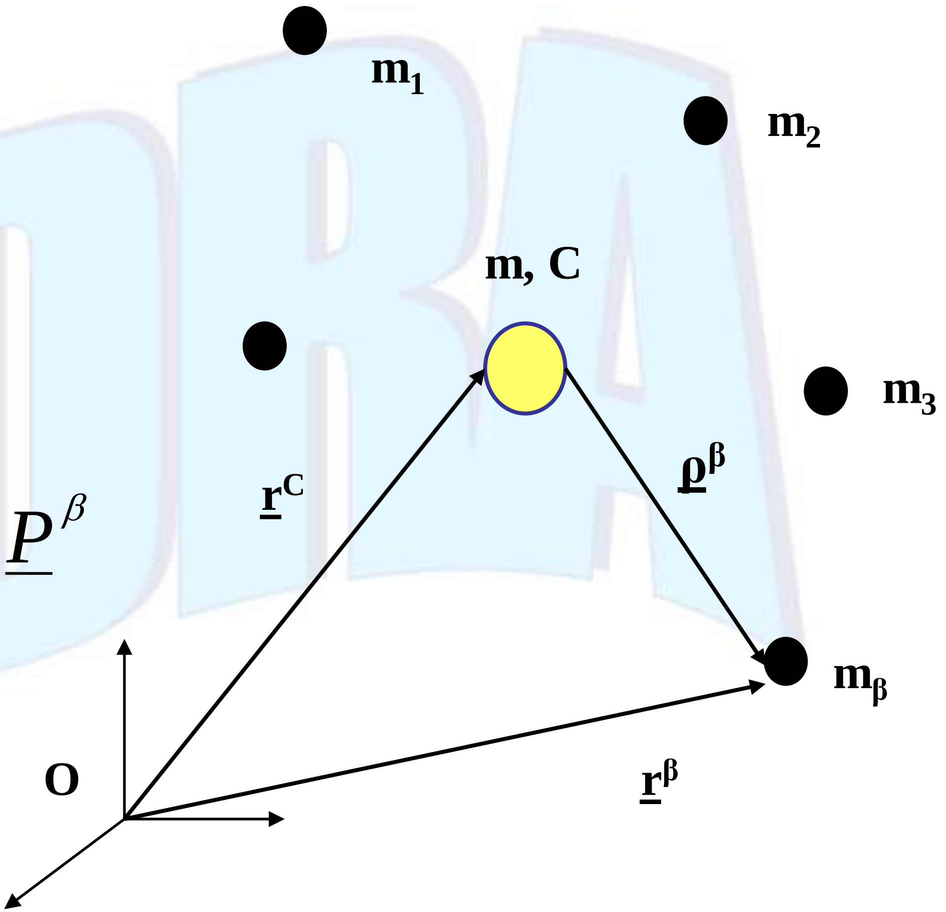
For a system of particles, by definition, we have:

$$\underline{H}^{O} = \sum_{\beta=1}^{N} \underline{r}^{\beta} \times \underline{P}^{\beta} =$$

$$= \sum_{\beta=1}^{N} (\underline{r}^{C} + \underline{\rho}^{\beta}) \times \underline{P}^{\beta}$$

$$= \sum_{\beta=1}^{N} \underline{\rho}^{\beta} \times \underline{P}^{\beta} + \underline{r}^{C} \times \sum_{\beta=1}^{N} \underline{P}^{\beta}$$

$$\underline{H}^{O} = \underline{H}^{C} + \underline{r}^{C} \times \underline{P}$$



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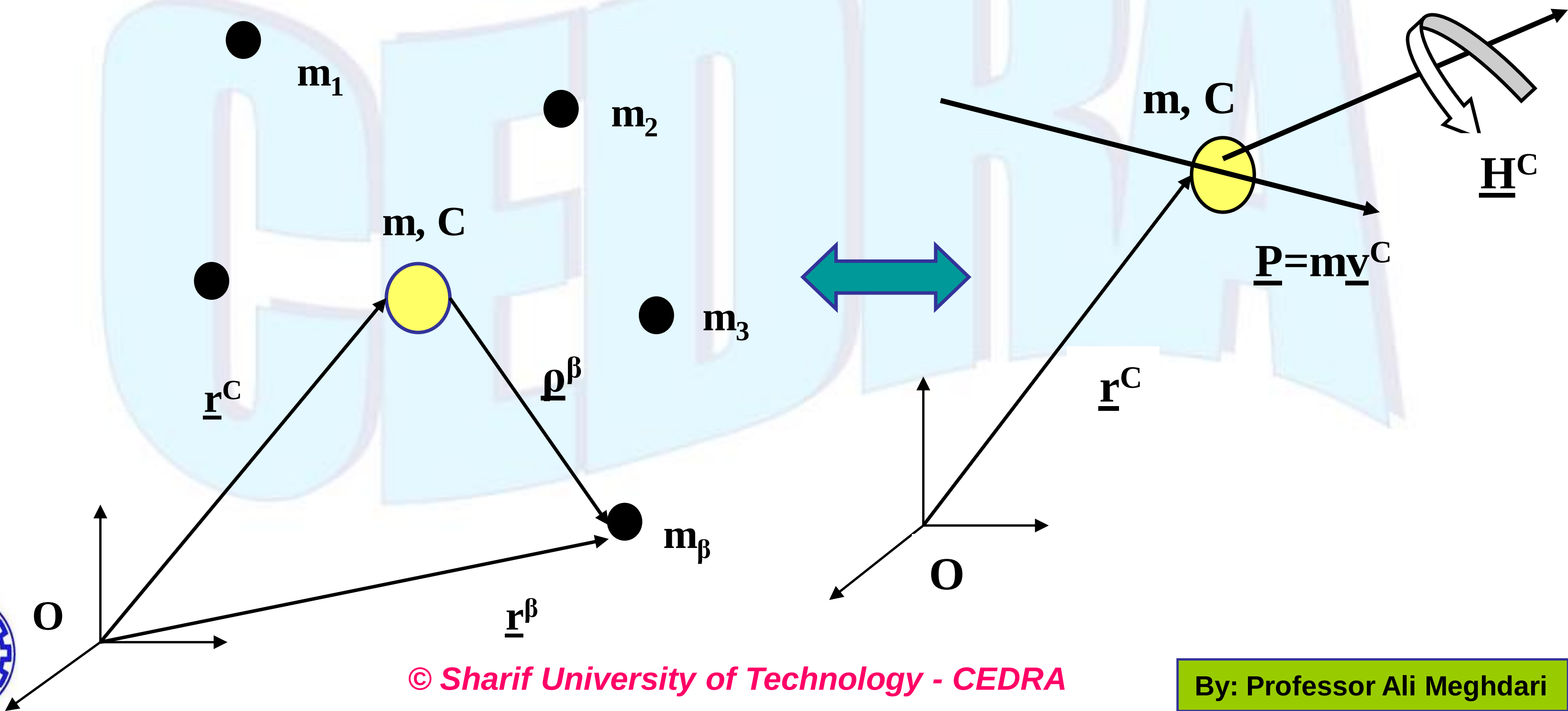
$$\underline{H}^{O} = \underline{H}^{C} + \underline{r}^{C} \times \underline{P} \quad \textbf{(Global M.O.M.)} \qquad (6.18)$$

Where; $\underline{H}^{O}$**: (M.O.M. about any point "O" in space).**

$$\underline{H}^{C} = \sum_{\beta=1}^{N} \underline{\rho}^{\beta} \times \underline{P}^{\beta} = \sum_{\beta=1}^{N} \underline{\rho}^{\beta} \times m_{\beta}\, \underline{\dot{\rho}}^{\beta}$$

(Central M.O.M. = M.O.M. about the mass-center)



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Theorem-17: **The central moment of momentum can be measured in any translatory reference frame fixed at the mass center.**

$$\underline{H}^C = \sum_{\beta=1}^{N} \underline{\rho}^\beta \times \underline{P}^\beta = \sum_{\beta=1}^{N} \underline{\rho}^\beta \times m_\beta \underline{\dot{\rho}}^\beta \qquad \textbf{(6.19)}$$

Proof:

since $\underline{r}^\beta = \underline{r}^C + \underline{\rho}^\beta \quad \rightarrow \quad \underline{v}^\beta = \underline{v}^C + \underline{\dot{\rho}}^\beta$ **, then:**

$$\underline{H}^C = \sum_{\beta=1}^{N} \underline{\rho}^\beta \times m_\beta \underline{v}^\beta = \sum_{\beta=1}^{N} m_\beta \underline{\rho}^\beta \times (\underline{\dot{\rho}}^\beta + \underline{v}^C) =$$

$$= \sum_{\beta=1}^{N} (m_\beta \underline{\rho}^\beta \times \underline{\dot{\rho}}^\beta) + (\underbrace{\sum_{\beta=1}^{N} m_\beta \underline{\rho}^\beta}_{0}) \times \underline{v}^C =$$

$$= \sum_{\beta=1}^{N} \underline{\rho}^\beta \times m_\beta \underline{\dot{\rho}}^\beta$$

The Kinetic Principles in the Newtonian Reference Frame (NRF):

The kinetic state of a material system is conserved unless disturbed by exterior actions. The *Principle of Momentum* (P.M.) and the *Principle of Moment of Momentum* (P.M.M) govern the change.

Recall: Newtonian (Inertial) Reference Frame; *Non-accelerating* & *Irrotational* reference frame.

***Admissible Newtonian Forces* in the NRF shown on a *Free-Body-Diagram* are:**

- **Contact Forces**
- **Field Forces (i.e. gravitational field, and electromagnetic field)**
- **Spring, and Friction Forces**


متشکرم