بسم الله الرحمن الرحیم

By: Professor Ali Meghdari

KINEMATICS (MOVING) REFERENCE FRAME

Purpose:

- **To Study Kinematics in a *Moving Reference Frame*.**

Topics:

- **Relative Motion.**
- **Moving (Kinematics) Reference Frame (i.e. *KRF*, or *MRF*).**

Absolute (i.e. Fixed or Primary) Kinematical Quantities*:** Quantities described in a ***Fixed (***Absolute***) ***reference frame***. (i.e. Absolute Velocity & Acceleration).

Relative Kinematical Quantities*:** Quantities described in a ***Moving (***Kinematics /Rigid Body***) ***reference frame***. (i.e. Relative Velocity & Acceleration).

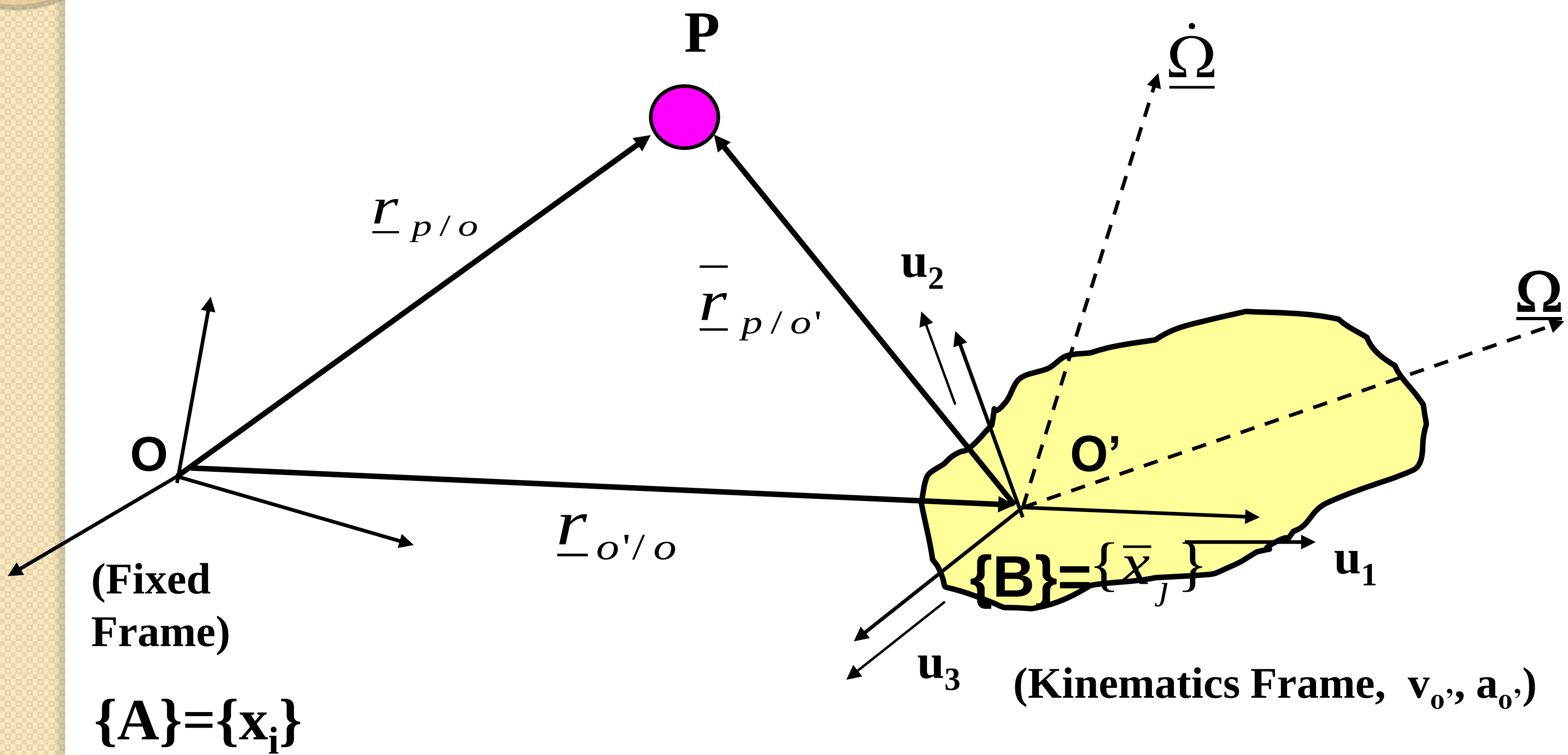
***Absolute and Relative Kinematical Quantities*:**
Let us consider the motion of a particle in space, in two reference frames such that:

{A}= $\{X_i\}$**:** ***Absolute*** (***Fixed/Primary***) ***Reference Frame***, **and**

{B}= $\{\overline{x}_j\}$**:** ***Moving*** (***Kinematics/Rigid Body***) ***Reference Frame***.

$\{A\}=\{x_i\}$: ***Absolute*** **(*Fixed/Primary*)** ***Reference Frame*****, and**

$\{B\}=\{\bar{x}_j\}$: ***Moving*** **(*Kinematics/Rigid Body*)** ***Reference Frame*****.**



Theorem-13: **The complete description of motion of a *kinematics reference frame* as rigid body consists of;**

1. The motion of the reference point "O′", as:. $\underline{v}_{O'}, \underline{a}_{O'}$

2. The angular Motion of the *KRF*, as: $\underline{\Omega}, \underline{\dot{\Omega}}$

The *relative motion* of the particle "P" with respect to the *KRF* "{B}" represented by a Cartesian coordinate set $\{\bar{x}_j\}$ **with unit vectors {$\underline{u}_j$} is:**

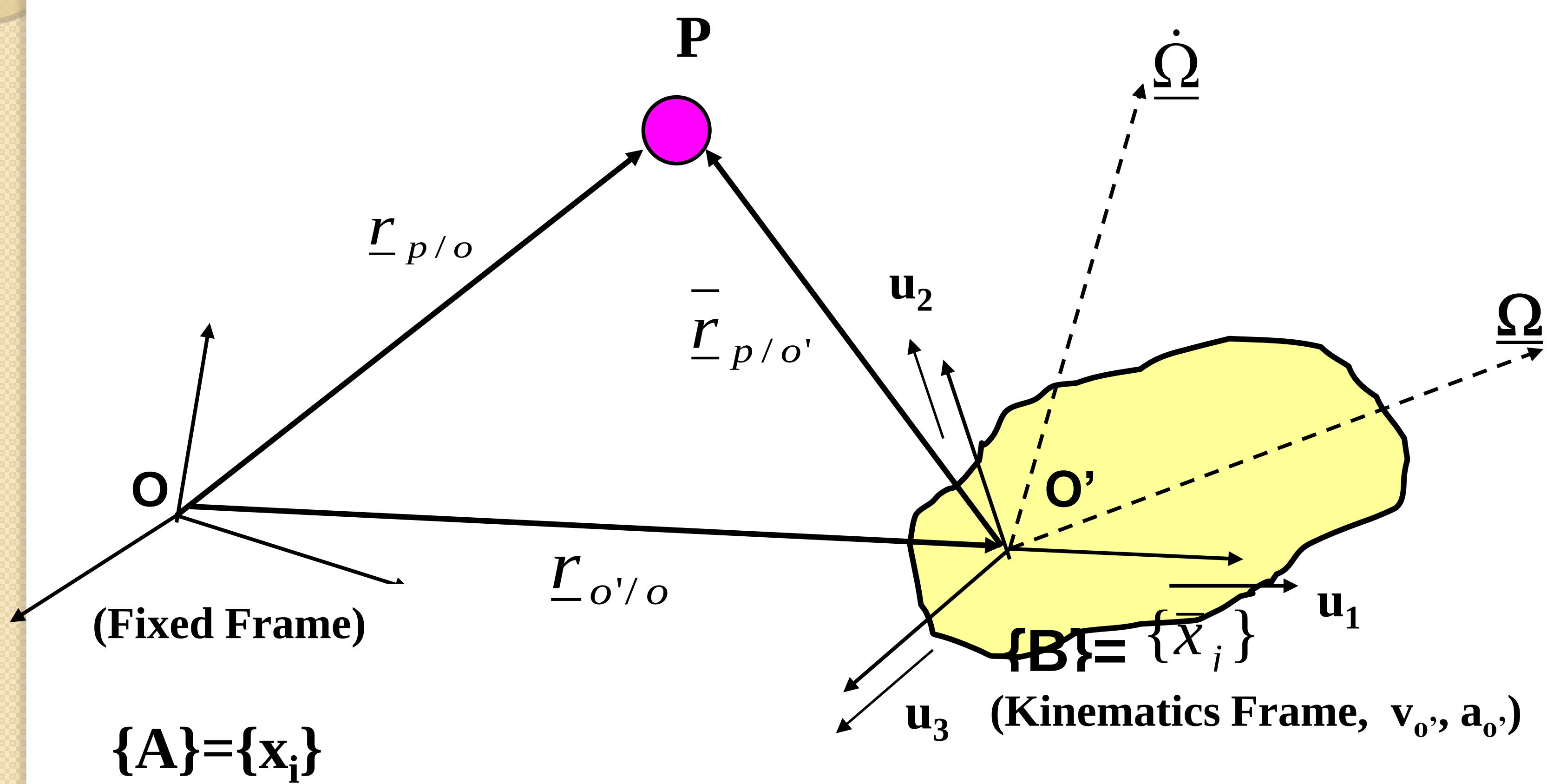
$$(\underline{r}_{P/O'})_{rel.} = \underline{\bar{r}}_{P/O'} = \bar{x}_j \underline{u}_j$$

$$(\underline{v}_{P/O'})_{rel.} = \underline{\bar{v}}_{P/O'} = \dot{\bar{x}}_j \underline{u}_j$$ **(if $\underline{u}_j$ are constant vectors)**

$$(\underline{a}_{P/O'})_{rel.} = \underline{\bar{a}}_{P/O'} = \ddot{\bar{x}}_j \underline{u}_j$$ **(if $\underline{u}_j$ are constant vectors)**

{A}=$\{x_i\}$: *Absolute* (*Fixed/Primary*) *Reference Frame*, and

{B}= $\{\bar{x}_j\}$: *Moving* (*Kinematics/Rigid Body*) *Reference Frame*.



By: Professor Ali Meghdari

Particle's Absolute Position:

$$\underline{r}_{P/O} = \underline{r}_{O'/O} + \underline{\bar{r}}_{P/O'} \qquad (5.1)$$

Particle's Absolute Velocity:

$$\underline{\dot{r}}_{P/O} = \underline{\dot{r}}_{O'/O} + \frac{d}{dt}(\underline{\bar{r}}_{P/O'})$$

Since "O" is a fixed point: $\underline{\dot{r}}_{P/O} = \underline{v}_P$ *and* $\underline{\dot{r}}_{O'/O} = \underline{v}_{O'}$

and using *the Jaumann Rate*:

$$\frac{d}{dt}(\underline{\bar{r}}_{P/O'}) = \dot{\bar{x}}_j \underline{u}_j + \underline{\Omega} \times \underline{\bar{r}}_{P/O'} = \underline{\bar{v}}_{P/O'} + \underline{\Omega} \times \underline{\bar{r}}_{P/O'}$$

Therefore :

$$\underline{v}_P = \underline{v}_{O'} + \underline{\bar{v}}_{P/O'} + \underline{\Omega} \times \underline{\bar{r}}_{P/O'} \qquad (5.2)$$

Particle's Absolute Acceleration:

$$\underline{\dot{v}}_P = \underline{\dot{v}}_{O'} + \frac{d}{dt}(\underline{\bar{v}}_{P/O'}) + \frac{d}{dt}(\underline{\Omega} \times \underline{\bar{r}}_{P/O'})$$

where;

$$\frac{d}{dt}(\underline{\Omega} \times \underline{\bar{r}}_{P/O'}) = \underline{\dot{\Omega}} \times \underline{\bar{r}}_{P/O'} + \underline{\Omega} \times \frac{d}{dt}(\underline{\bar{r}}_{P/O'}) = \underline{\dot{\Omega}} \times \underline{\bar{r}}_{P/O'} + \underline{\Omega} \times \underline{\bar{v}}_{P/O'} + \underline{\Omega} \times (\underline{\Omega} \times \underline{\bar{r}}_{P/O'})$$

$$\frac{d}{dt}(\underline{\bar{v}}_{P/O'}) = \ddot{\bar{x}}_j \underline{u}_j + \underline{\Omega} \times \underline{\bar{v}}_{P/O'} = \underline{\bar{a}}_{P/O'} + \underline{\Omega} \times \underline{\bar{v}}_{P/O'}$$

$$\underline{a}_P = \underline{a}_{O'} + \underline{\bar{a}}_{P/O'} + \underline{\dot{\Omega}} \times \underline{\bar{r}}_{P/O'} + 2\underline{\Omega} \times \underline{\bar{v}}_{P/O'} + \underline{\Omega} \times (\underline{\Omega} \times \underline{\bar{r}}_{P/O'}) \quad \textbf{(5.3)}$$

Abs. Acc. Of "P"	**Abs. Acc. of center of *KRF* "O´"**	**Acc. of "P" observed in *KRF***	**Euler's (tangential) Acc.**	**Coriolis Acc.**	**Centripetal Acc.**

Special Cases:

1. If *KRF* is ***Translatory*** (***Irrotational***) ***Reference Frame***, then $(\underline{\Omega}=0, and\, \dot{\underline{\Omega}}=0)$:

$$\underline{v}_P=\underline{v}_{O'}+\overline{\underline{v}}_{P/O'} \qquad (5.4)$$

$$\underline{a}_P=\underline{a}_{O'}+\overline{\underline{a}}_{P/O'} \qquad (5.5)$$

2. If the ***Center of KRF is a Fixed Point***, then $(\underline{v}_{O'}=0, and\, \underline{a}_{O'}=0)$:

$$\underline{v}_P=\overline{\underline{v}}_{P/O'}+\underline{\Omega}\times\overline{\underline{r}}_{P/O'} \qquad (5.6)$$

$$\underline{a}_P=\overline{\underline{a}}_{P/O'}+\dot{\underline{\Omega}}\times\overline{\underline{r}}_{P/O'}+2\underline{\Omega}\times\overline{\underline{v}}_{P/O'}+\underline{\Omega}\times(\underline{\Omega}\times\overline{\underline{r}}_{P/O'}) \qquad (5.7)$$

3. If the *Point* P *is a Fixed Point in the KRF*, then:

$$(\overline{\underline{v}}_{P/O'} = 0, and\, \overline{\underline{a}}_{P/O'} = 0)$$

In this case point P is on a rigid body, and since *KRF* is attached to the rigid body, therefore:

$$(\underline{\Omega} = \underline{\omega}_{R.B.} = \underline{\omega}, and \qquad \dot{\underline{\Omega}} = \underline{\alpha}_{R.B.} = \underline{\alpha})$$ **, and**

$$\underline{v}_P = \underline{v}_{O'} + \underline{\Omega} \times \overline{\underline{r}}_{P/O'} = \underline{v}_{O'} + \underline{\omega} \times \overline{\underline{r}}_{P/O'} \qquad \boxed{(5.8)}$$

$$\underline{a}_P = \underline{a}_{O'} + \dot{\underline{\Omega}} \times \overline{\underline{r}}_{P/O'} + \underline{\Omega} \times (\underline{\Omega} \times \overline{\underline{r}}_{P/O'}) = \underline{a}_{O'} + \underline{\alpha} \times \overline{\underline{r}}_{P/O'} + \underline{\omega} \times (\underline{\omega} \times \overline{\underline{r}}_{P/O'})$$

$$\boxed{(5.9)}$$

To study relative motion of a particle P as observed in *KRF*, we need:

$\Gamma_{rel.}$: ***Relative Path***

$$(\underline{r}_{P/O'})_{rel.} = \overline{\underline{r}}_{P/O'}$$: ***Relative Position of* P *in the KRF.***

$$(\underline{v}_{P/O'})_{rel.} = \overline{\underline{v}}_{P/O'}$$: ***Relative Velocity of* P *in the KRF.***

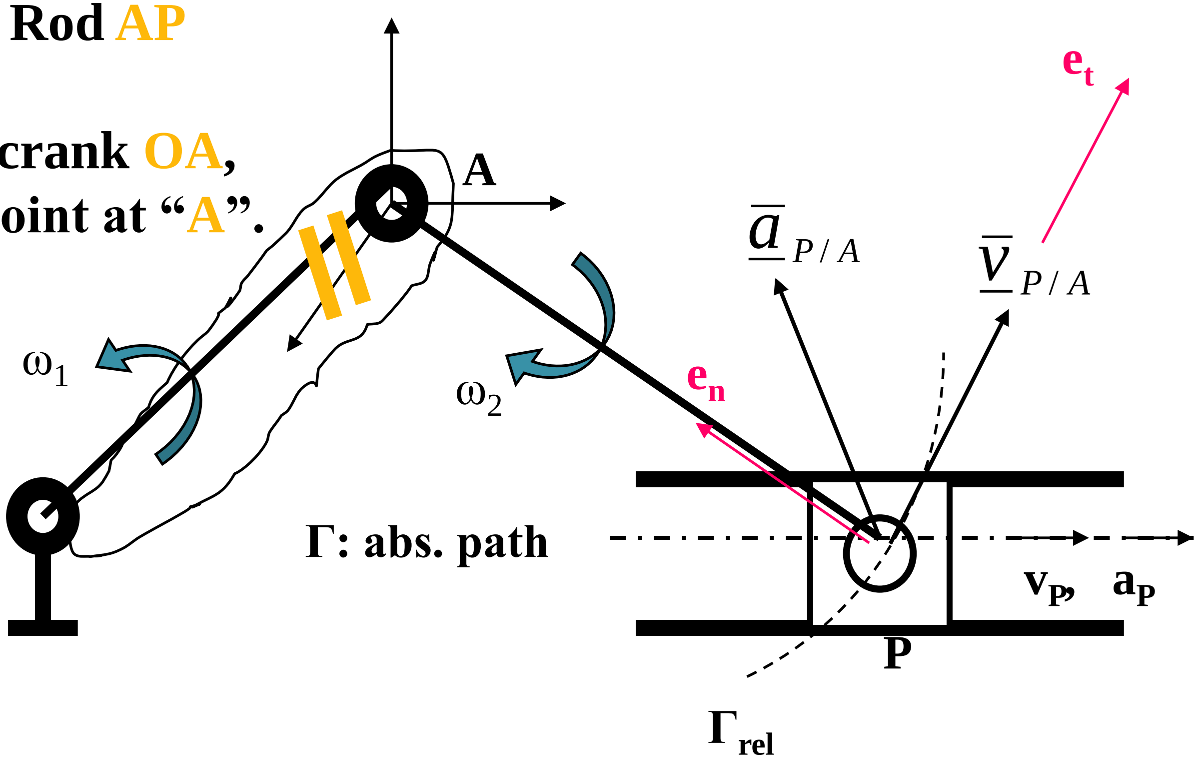
$$(\underline{a}_{P/O'})_{rel.} = \overline{\underline{a}}_{P/O'}$$: ***Relative Acceleration of* P *in the KRF.***

Example: Consider the Slider-Crank Mechanism shown:

Absolute Reference Frame: **Black Board**
Kinematics Reference Frame:

- **The Crank OA, or**
- **The Connecting Rod AP**

1. Set *KRF* on the crank OA, with reference point at "A".



a). ***KRF Motion*** :

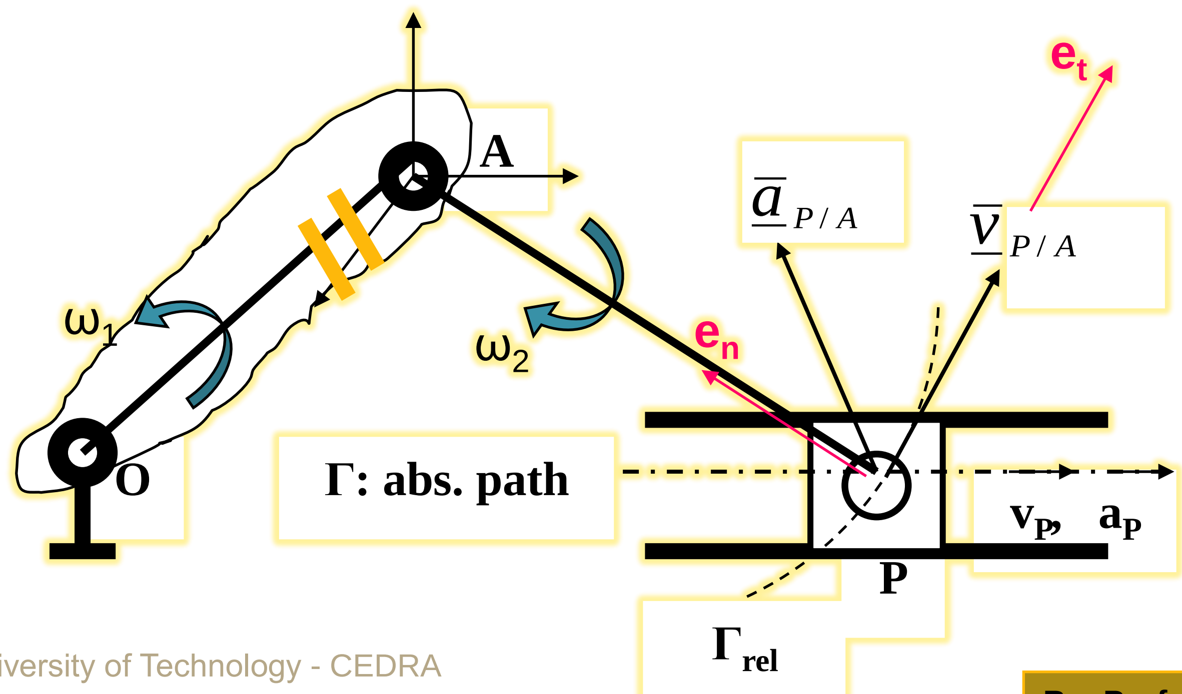
$$\underline{v}_A = ?, \underline{a}_A = ?, \underline{\omega}_{OA} = \underline{\omega}_{KRF} = \omega_1 \underline{e}_3, \underline{\alpha}_{OA} = \underline{\alpha}_{KRF} = \alpha_1 \underline{e}_3$$

b). ***Relative Motion of* P** :

$$\underline{\bar{r}}_{P/A} = ?, \underline{\bar{v}}_{P/A} = ?, \underline{\bar{a}}_{P/A} = ?, given : \underline{\omega}_{AP}, and \, \underline{\alpha}_{AP}$$

c). ***Absolute Motion of* P** :

$$\underline{v}_P = ...?, and \qquad \underline{a}_P = ...?$$



By: Professor Ali Meghdari

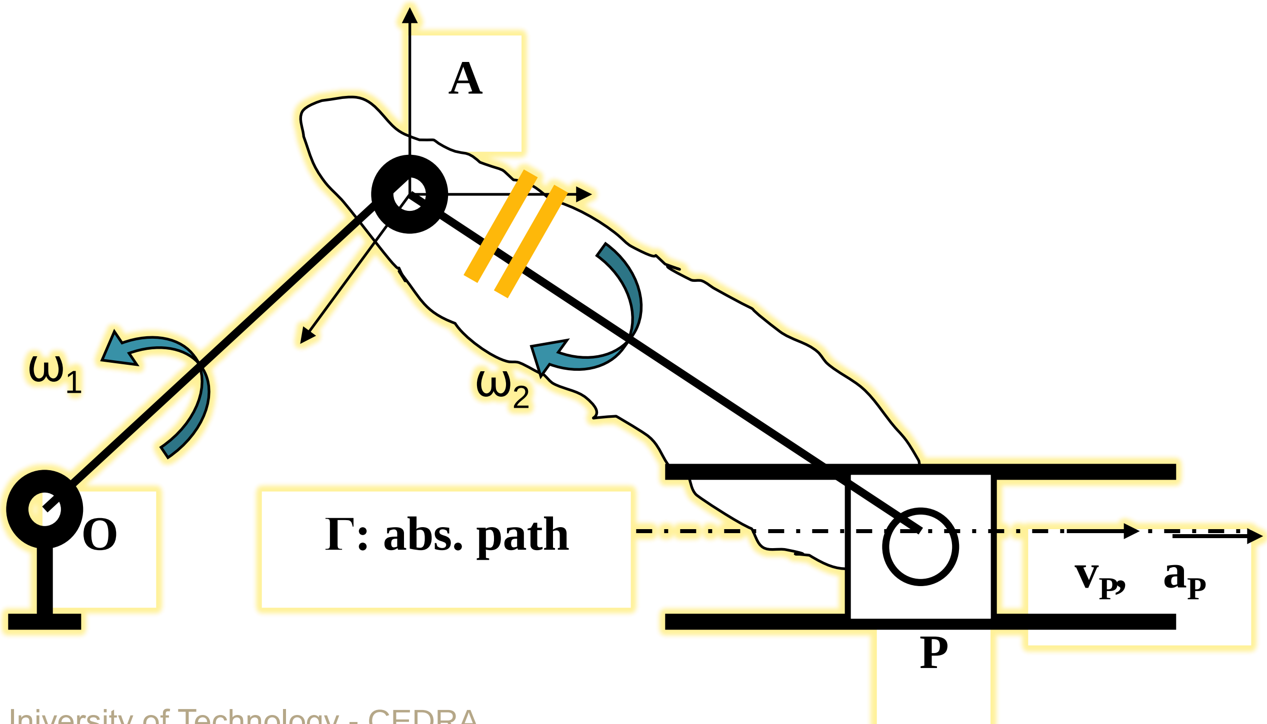
2. Set **KRF** on the rod **AP**, with reference point at "**A**".

a). ***KRF Motion*** :

$$\underline{v}_A = ?, \underline{a}_A = ?, \underline{\omega}_{AP} = \underline{\omega}_{KRF} = (\omega_1 - \omega_2)\underline{e}_3, \underline{\alpha}_{AP} = \underline{\alpha}_{KRF} = (\alpha_1 - \alpha_2)\underline{e}_3$$

b). ***Relative Motion of* P**: $(no-relative-motion) \rightarrow \underline{\bar{v}}_{P/A} = 0, \underline{\bar{a}}_{P/A} = 0$

c). ***Absolute Motion of* P**: $\underline{v}_P = ...?\, and \qquad \underline{a}_P = ...?$



Types of Kinematics Reference Frame Problems:

Class of Problems	Absolute Quantity	KRF Motion	Relative Quantity
I	? ?	√ √	√ √
II	√ √	? ?	√ √
III	√ ?	√ √	√ ?
Most Difficult Problems IV $\begin{Bmatrix}(a)\\(b)\end{Bmatrix}$	√ √ √ ?	√ ? √ ?	√ ? √ √

Rigid Body Motion (using KRF):

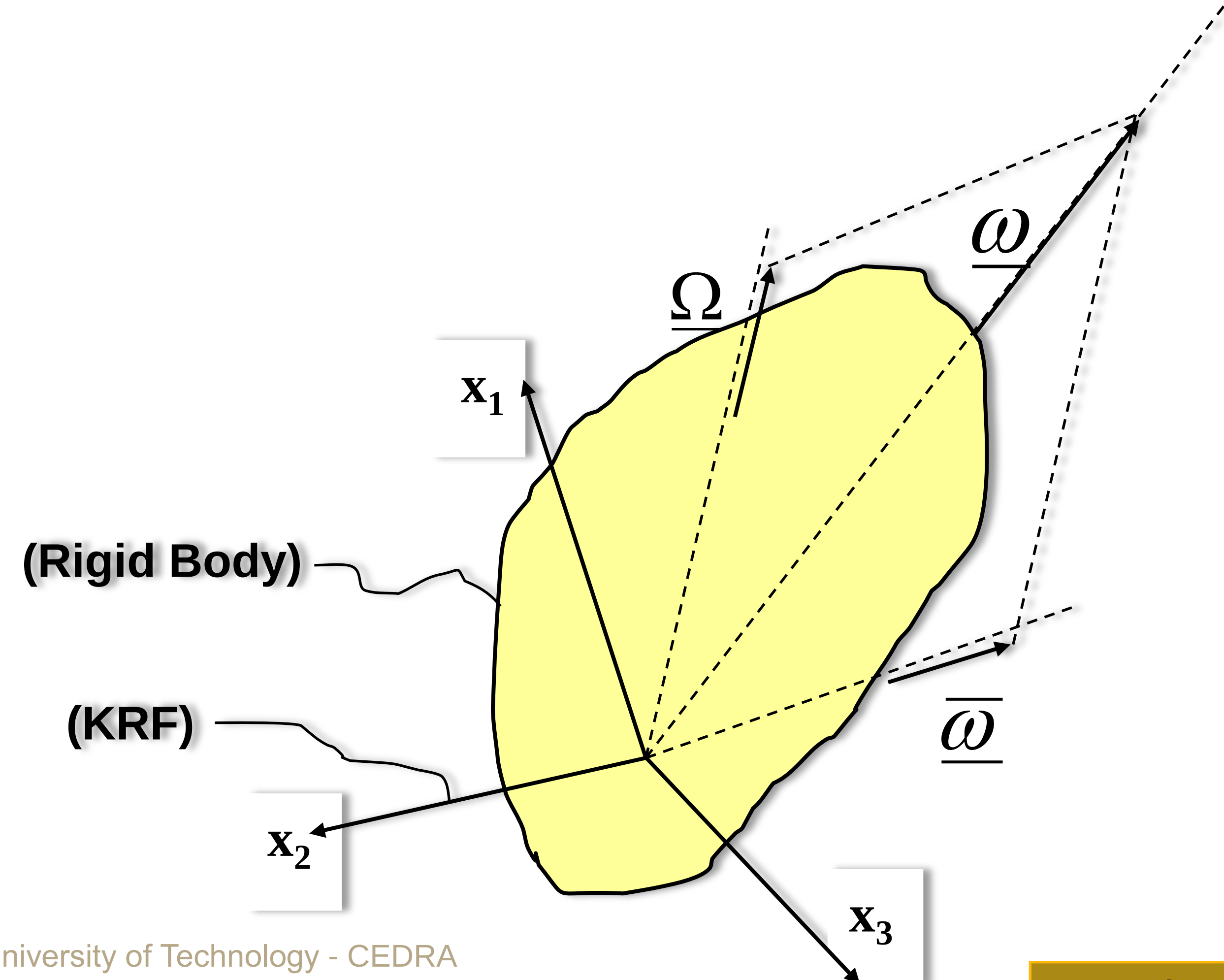
Theorem-14: **The angular velocity of a Rigid Body "$\underline{\omega}$" is the vector sum of the angular velocity of the *KRF* "$\underline{\Omega}$" and the relative angular velocity of the rigid body in the *KRF* "$\underline{\overline{\omega}}$".**

$$\underline{\omega} = \underline{\Omega} + \underline{\overline{\omega}} \qquad (5.10)$$

Proof: **{By simultaneous rotation of the rigid body about two axis}.**

$$\underline{\omega} = \underline{\Omega} + \underline{\overline{\omega}} \tag{5.10}$$

***Proof*: {By simultaneous rotation of the rigid body about two axis}.**



Theorem-15: **The angular acceleration of a rigid body "$\underline{\alpha}$" is related to the coordinate angular acceleration "$\dot{\underline{\Omega}}$", the relative angular acceleration "$\underline{\overline{\alpha}}$", as well as the angular velocity properties.**

$$\underline{\alpha} = \frac{d\underline{\omega}}{dt} = \dot{\underline{\Omega}} + \frac{d}{dt}\underline{\overline{\omega}} = \dot{\underline{\Omega}} + (\dot{\overline{\omega}}_i \underline{u}_i + \underline{\Omega} \times \underline{\overline{\omega}}) = \dot{\underline{\Omega}} + \overline{\alpha}_i \underline{u}_i + \underline{\Omega} \times \underline{\overline{\omega}}$$

$$\underline{\alpha} = \dot{\underline{\Omega}} + \underline{\overline{\alpha}} + \underline{\Omega} \times \underline{\overline{\omega}} \qquad \textbf{(5.11)}$$

Example: **A radar antenna rotates about a fixed vertical axis at a constant angular velocity "ω_0", and the angle "θ" oscillates at $\theta = a_0 + a_1 \sin \omega_1 t$, where "θ" is in radians and "t" is in seconds. Determine the velocity and acceleration of the probe P using the moving coordinate system {$\bar{X}_i$} attached to:**

(a). the vertical shaft?

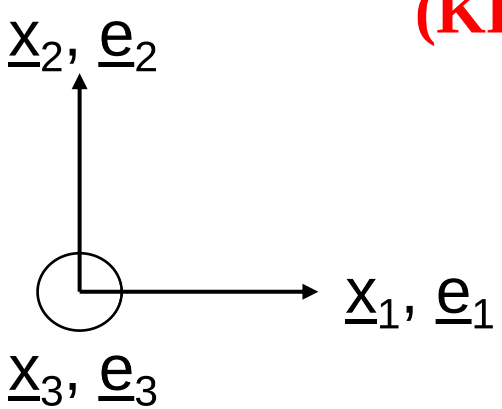
(b). the antenna?

Solution: Given;

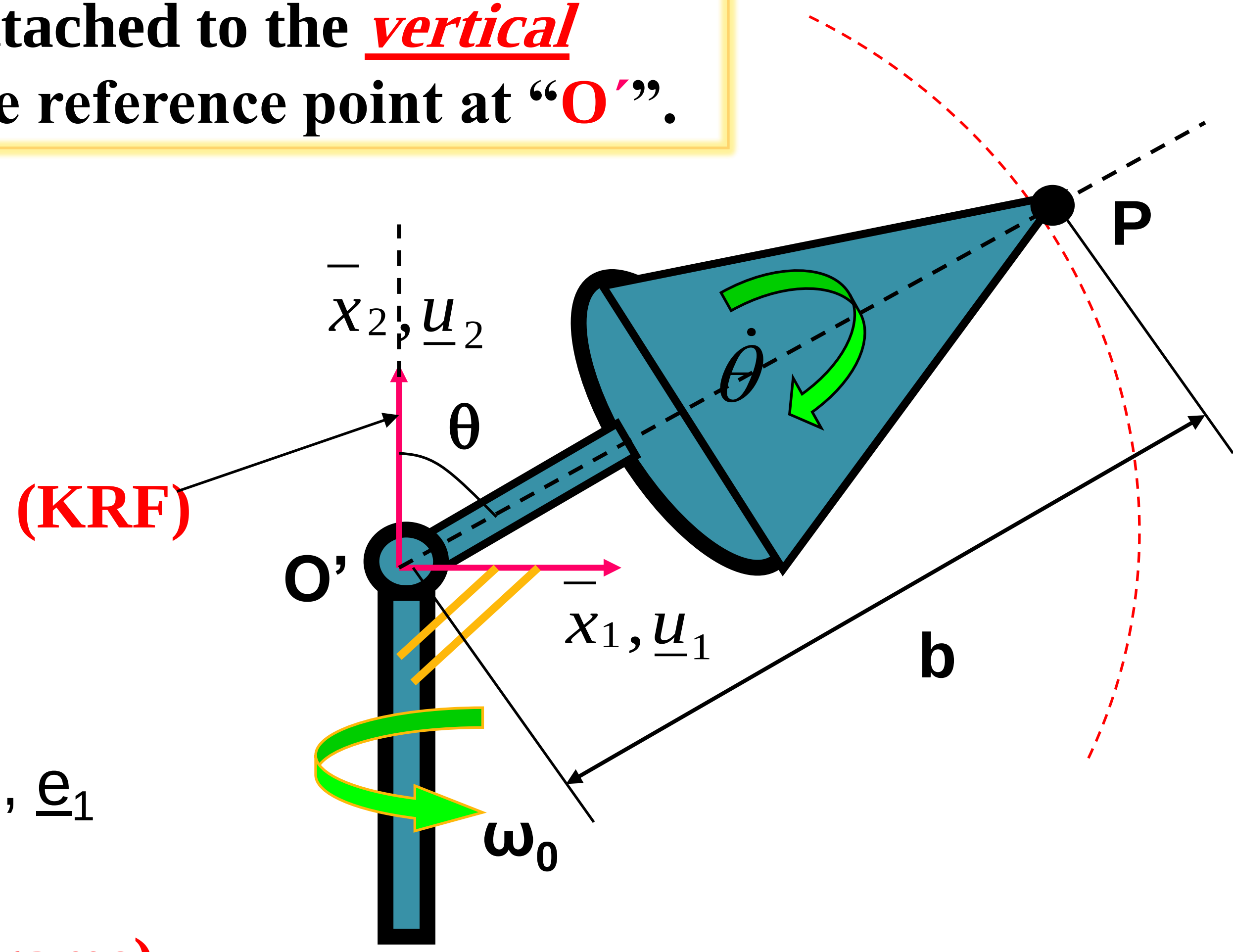
$$\begin{cases} \omega_0 = const. \rightarrow \alpha_0 = 0 \\ \theta = a_0 + a_1 \sin \omega_1 t \end{cases}$$

Find: $\underline{v}_P = ?, \quad \underline{a}_P = ?$

(a). *KRF* is attached to the *vertical shaft*, with the reference point at "O′".



(Abs. Ref. Frame)



By: Professor Ali Meghdari

1. ***KRF Motion***:

$$\underline{\Omega} = \omega_0 \underline{u}_2 = \omega_0 \underline{e}_2 , \quad \dot{\underline{\Omega}} = 0, \quad \underline{v}_{O'} = 0, \quad \underline{a}_{O'} = 0$$

2. ***Relative Quantities***:

$$\bar{\underline{r}}_{P/O'} = b(\sin\theta \underline{u}_1 + \cos\theta \underline{u}_2)$$

$$\bar{\underline{v}}_{P/O'} = \frac{d}{dt} \bar{\underline{r}}_{P/O'} = (-\dot{\theta} \underline{u}_3) \times \bar{\underline{r}}_{P/O'} = b\dot{\theta}\cos\theta \underline{u}_1 - b\dot{\theta}\sin\theta \underline{u}_2$$

$$\bar{\underline{a}}_{P/O'} = \frac{d}{dt} \bar{\underline{v}}_{P/O'} = b(\ddot{\theta}\cos\theta - \dot{\theta}^2 \sin\theta)\underline{u}_1 - b(\ddot{\theta}\sin\theta + \dot{\theta}^2 \cos\theta)\underline{u}$$

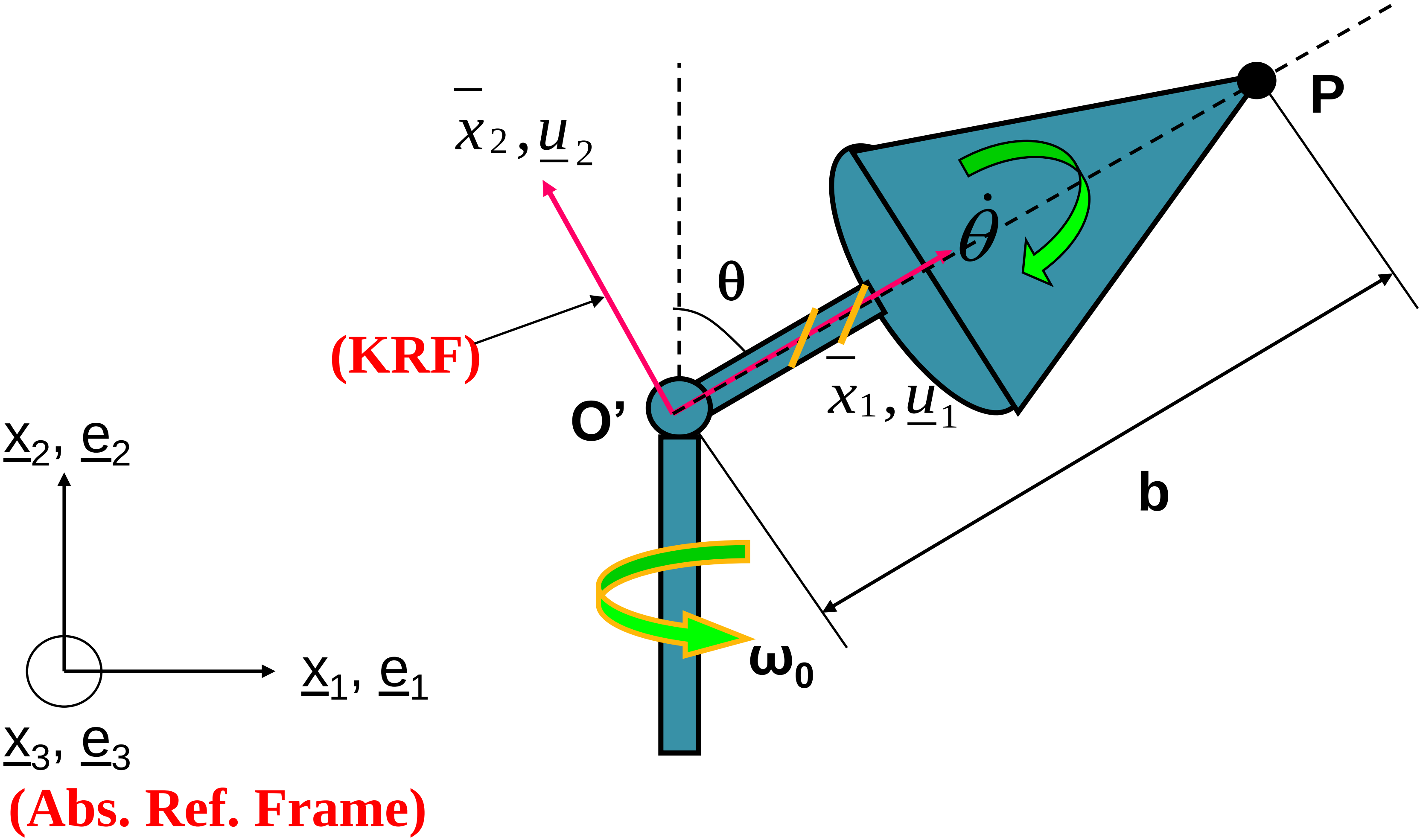
3. ***Absolute Quantities:***

$$\begin{cases} \underline{v}_P = \underline{v}_{O'} + \underline{\bar{v}}_{P/O'} + \underline{\Omega} \times \underline{\bar{r}}_{P/O'} \\ \underline{v}_P = b\dot{\theta}\cos\theta \underline{u}_1 - b\dot{\theta}\sin\theta \underline{u}_2 + (\omega_0 \underline{u}_2) \times \underline{\bar{r}}_{P/O'} \\ \underline{v}_P = b\dot{\theta}\cos\theta \underline{u}_1 - b\dot{\theta}\sin\theta \underline{u}_2 - b\omega_0 \sin\theta \underline{u}_3 \quad , \quad \dot{\theta} = a_1\omega_1 \cos\omega_1 t \end{cases}$$

$$\begin{cases} \underline{a}_P = \underbrace{\cancel{\underline{a}_{O'}}}_{0} + \underline{\bar{a}}_{P/O'} + \underbrace{\cancel{\dot{\underline{\Omega}}}}_{0} \times \underline{\bar{r}}_{P/O'} + 2\underline{\Omega} \times \underline{\bar{v}}_{P/O'} + \underline{\Omega} \times (\underline{\Omega} \times \underline{\bar{r}}_{P/O'}) \\ \underline{a}_P = \underline{\bar{a}}_{P/O'} - 2b\omega_0\dot{\theta}\cos\theta \underline{u}_3 - b\omega_0^2 \sin\theta \underline{u}_1 \\ \underline{a}_P = b(\ddot{\theta}\cos\theta - \dot{\theta}^2 \sin\theta - \omega_0^2 \sin\theta)\underline{u}_1 - b(\ddot{\theta}\sin\theta + \dot{\theta}^2 \cos\theta)\underline{u}_2 - 2b\omega_0\dot{\theta}\cos\theta \underline{u}_3 \end{cases}$$

$$\ddot{\theta} = -a_1\omega_1^2 \sin\omega_1 t$$

(b). ***KRF*** **is attached to the *antenna* with the reference point at "O′".**



1. *KRF Motion*:

$$\underline{\Omega} = \dot{\theta}(-\underline{u}_3) + \omega_0 \underline{e}_2 = -\dot{\theta}\underline{u}_3 + \omega_0(\cos\theta \underline{u}_1 + \sin\theta \underline{u}_2) \rightarrow$$

$$\underline{\Omega} = \omega_0 \cos\theta \underline{u}_1 + \omega_0 \sin\theta \underline{u}_2 - \dot{\theta}\underline{u}_3 \qquad , and$$

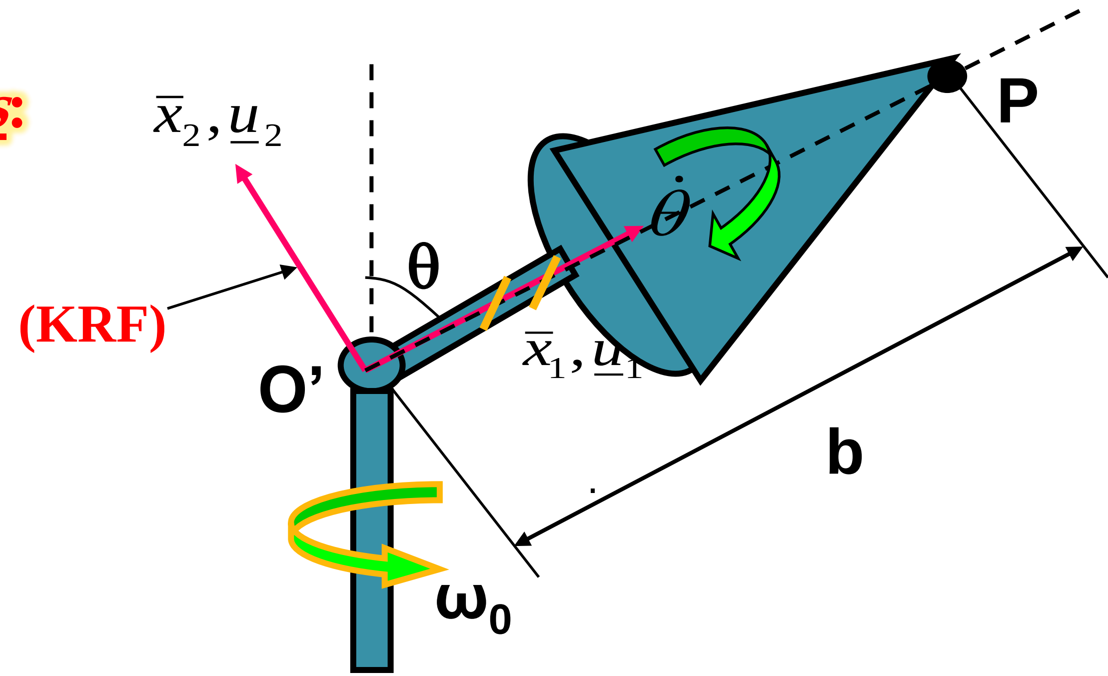
$$\dot{\underline{\Omega}} = -\dot{\theta}\omega_0 \sin\theta \underline{u}_1 + \dot{\theta}\omega_0 \cos\theta \underline{u}_2 - \ddot{\theta}\underline{u}_3 , and \quad \underline{v}_{O'} = 0, \quad \underline{a}_{O'} =$$

2. *Relative Quantities*:

$$\bar{r}_{P/O'} = b\underline{u}_1$$

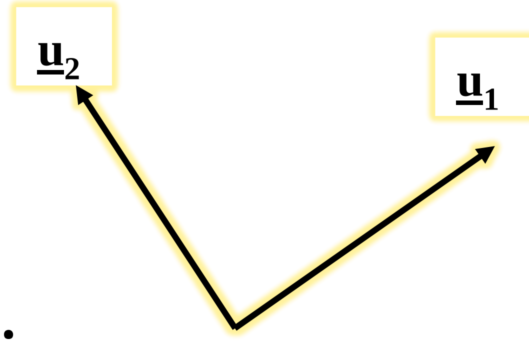
$$\bar{\underline{v}}_{P/O'} = 0$$

$$\bar{\underline{a}}_{P/O'} = 0$$



By: Professor Ali Meghdari

3. *Absolute Quantities*:



$$\begin{cases} \underline{v}_P = \cancelto{0}{\underline{v}_{O'}} + \cancelto{0}{\underline{\bar{v}}_{P/O'}} + \underline{\Omega} \times \underline{\bar{r}}_{P/O'} \\ \underline{v}_P = -b\dot{\theta}\underline{u}_2 - b\omega_0 \sin\theta \underline{u}_3 \end{cases} \quad , \quad \dot{\theta} = a_1\omega_1 \cos\omega_1 t$$

$$\begin{cases} \underline{a}_P = \cancelto{0}{\underline{a}_{O'}} + \cancelto{0}{\underline{\bar{a}}_{P/O'}} + \underline{\dot{\Omega}} \times \underline{\bar{r}}_{P/O'} + 2\underline{\Omega} \times \underline{\bar{v}}_{P/O'} + \underline{\Omega} \times (\underline{\Omega} \times \underline{\bar{r}}_{P/O'}) \\ \underline{a}_P = -b\omega_0\dot{\theta}\cos\theta\underline{u}_3 - b\ddot{\theta}\underline{u}_2 - b\omega_0\dot{\theta}\cos\theta\underline{u}_3 + b\omega_0^2\sin\theta\cos\theta\underline{u}_2 - b\omega_0^2\sin^2\theta\underline{u}_1 - b\dot{\theta}^2\underline{u}_1 \\ \underline{a}_P = b(-\dot{\theta}^2 - \omega_0^2\sin^2\theta)\underline{u}_1 - b(\ddot{\theta} - \omega_0^2\sin\theta\cos\theta)\underline{u}_2 - 2b\omega_0\dot{\theta}\cos\theta\underline{u}_3 \end{cases}$$

$$\ddot{\theta} = -a_1\omega_1^2 \sin\omega_1 t$$

متشکرم

By: Professor Ali Meghdari