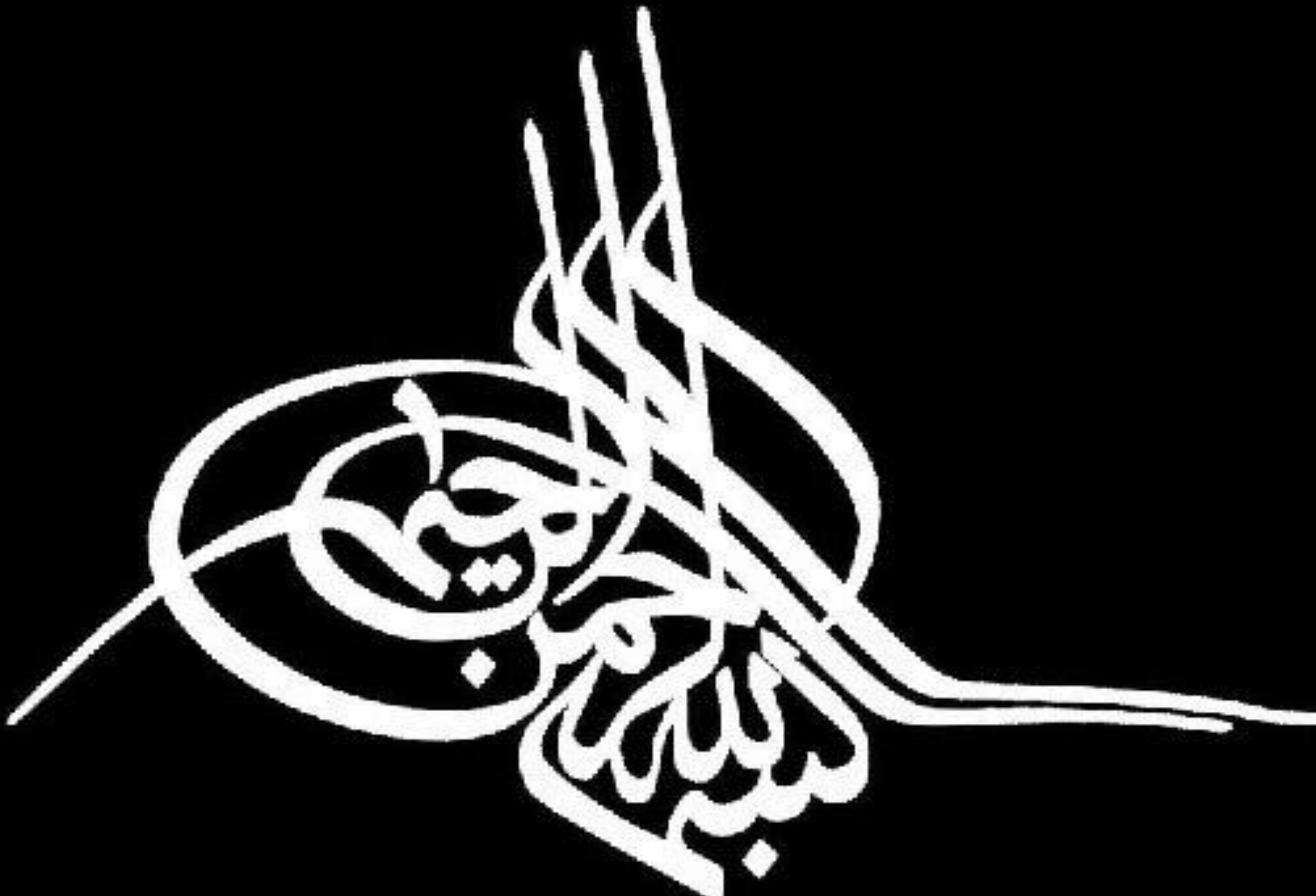

بسم الله الرحمن الرحيم

RIGID BODY KINEMATICS

Purpose:

- **Analytical Description of Rigid Body Motion.**
- **Matrix Transforms to Represent Rigid Body Motion.**
- **Reinforcement of Elementary Kinematical Equations.**

Topics:

- **Translation of Rigid Bodies.**
- **Rotation of Rigid Bodies.**
- **General Motion of Rigid Bodies (i.e. Robot Kinematics)**
- **Coordinate Transformations**

Rotation About an Arbitrary Axis (Equivalent Angle-Axis Representation):

Euler's Theorem(10-continued): Any change of orientation for a rigid body with a fixed body point can be accomplished through a *General Rotation Operator* (a simple rotation) with a proper axis and angle selection.

Consider the following coordinates:

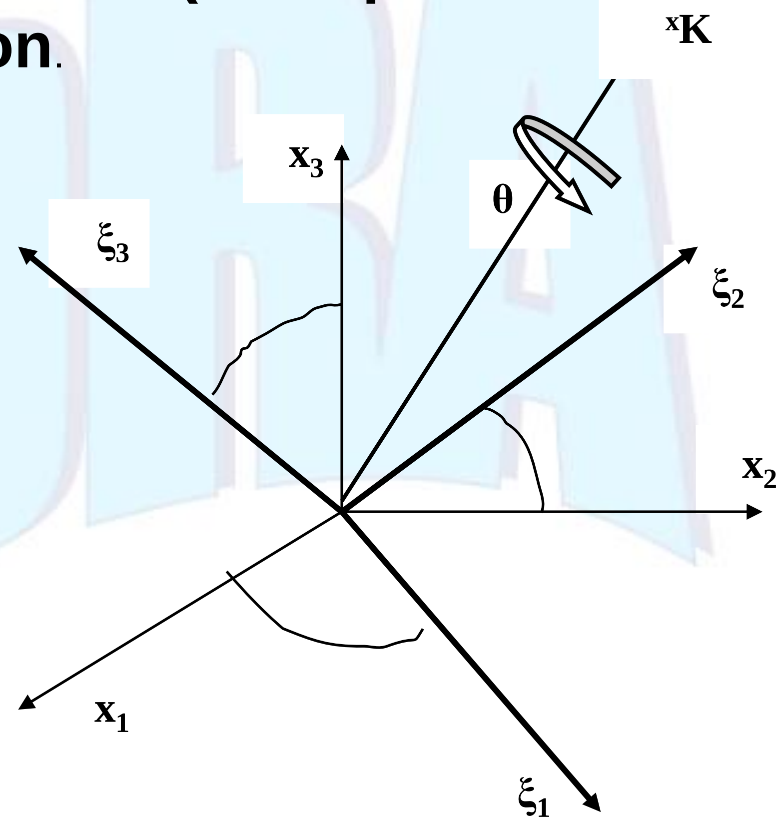
$\{x_i\}$: **Spatial Coordinates**

$\{\xi_j\}$: **Body Coordinates**

$$\{x_i\} = \underline{\underline{R}}\{\xi_j\}$$

$$\underline{\underline{R}} = \underline{\underline{R}}({}^x\underline{K},\theta) = {}^x_\xi\underline{\underline{R}}({}^x\underline{K},\theta)$$

= A Simple/General Rotation Operator about an arbitrary axis.



Where:

$$\underline{\underline{R}}({}^{x}\underline{K},\theta)=\begin{bmatrix} k_{x1}k_{x1}v\theta+c\theta & k_{x1}k_{x2}v\theta-k_{x3}s\theta & k_{x1}k_{x3}v\theta+k_{x2}s\theta \\ k_{x1}k_{x2}v\theta+k_{x3}s\theta & k_{x2}k_{x2}v\theta+c\theta & k_{x2}k_{x3}v\theta-k_{x1}s\theta \\ k_{x1}k_{x3}v\theta-k_{x2}s\theta & k_{x2}k_{x3}v\theta+k_{x1}s\theta & k_{x3}k_{x3}v\theta+c\theta \end{bmatrix}$$

And;

$${}^{x}\underline{K}=k_{x1}\underline{e}_1+k_{x2}\underline{e}_2+k_{x3}\underline{e}_3=\begin{bmatrix}k_{x1} & k_{x2} & k_{x3}\end{bmatrix}^t \quad and \quad k_{x1}^2+k_{x2}^2+k_{x3}^2=1$$

$$v\theta=vers\theta=(1-\cos\theta)$$

<u>Ex</u>:

$$\underline{\underline{R}}(x_1,\theta)=\begin{bmatrix}1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta\end{bmatrix} \quad is \quad \underline{\underline{R}}({}^{x}\underline{K},\theta) \quad where: k_{x1}=1,\quad k_{x2}=0,\quad k_{x3}=0$$

For a given Rotation Matrix like $\underline{\underline{R}} = {}^{x}_{\xi}\underline{\underline{R}}({}^{x}\underline{K},\theta) = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$**, one can**

Determine *__the equivalent angle-axis__* by taking an inverse approach, such that:

$$\sin\theta = \pm\frac{1}{2}\sqrt{(r_{32}-r_{23})^2 + (r_{13}-r_{31})^2 + (r_{21}-r_{12})^2}$$ **, and**

$$\cos\theta = \frac{r_{11}+r_{22}+r_{33}-1}{2}$$ **, where:**

$$\theta = \tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right) \qquad {}^{x}\underline{K} = \frac{1}{2\sin\theta}\begin{bmatrix} r_{32}-r_{23} \\ r_{13}-r_{31} \\ r_{21}-r_{12} \end{bmatrix} = \begin{bmatrix} k_{x1} \\ k_{x2} \\ k_{x3} \end{bmatrix}$$

This solution is valid for ($0 \prec \theta \prec 180$), and for every pair of equivalent angle-axis $({}^{x}\underline{K},\theta)$, there exists another pair as $(-{}^{x}\underline{K},-\theta)$ representing the same orientation in space with the same rotation matrix. (no solutions for θ=0 and 180).

Any combination of Rotations is always equivalent to a single rotation about some axis "K" by an angle "θ".

Ex: Let

$$\underline{\underline{R}} = \underline{\underline{R}}(x_2,90)\underline{\underline{R}}(x_3,90) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

From above equations we have:

$$\sin\theta = \pm\sqrt{(1-0)^2+(1-0)^2+(1-0)^2} = \pm\frac{\sqrt{3}}{2}$$

$$\cos\theta = \frac{0+0+0-1}{2} = \frac{-1}{2}$$

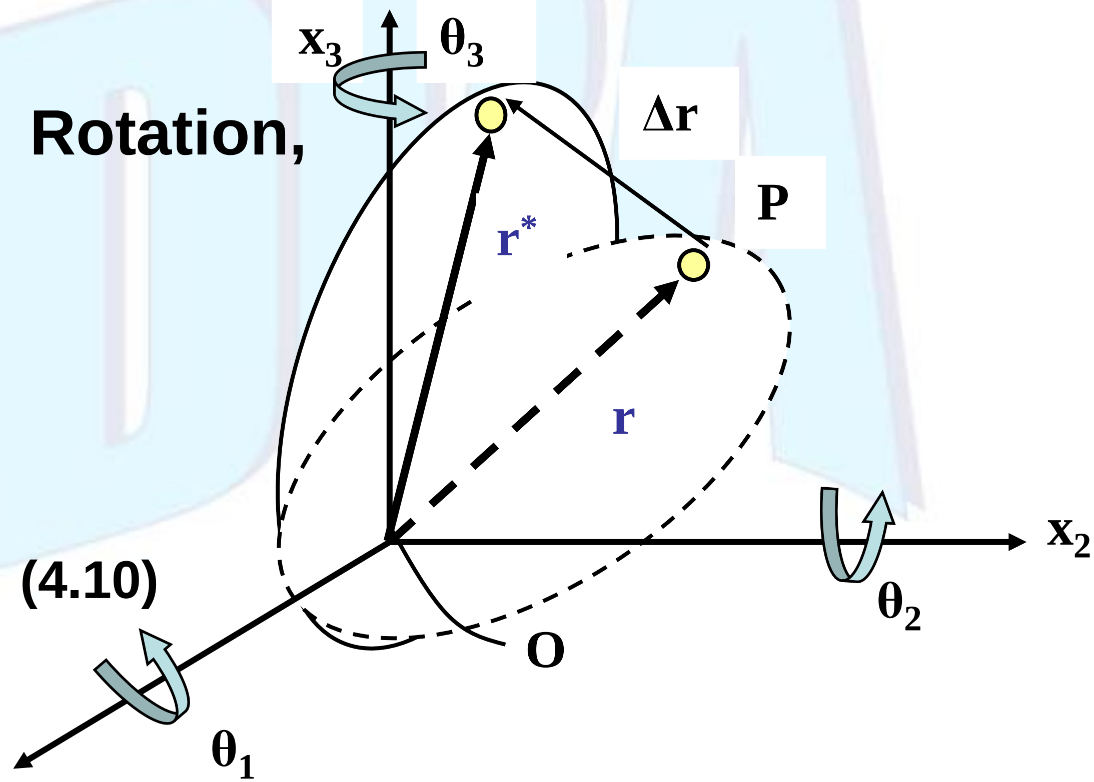
$$\theta = \tan^{-1}\left(\frac{\pm\sqrt{3}/2}{-1/2}\right) = \pm 120^\circ$$

$$\underline{K} = \frac{1}{\sqrt{3}}\underline{e}_1 + \frac{1}{\sqrt{3}}\underline{e}_2 + \frac{1}{\sqrt{3}}\underline{e}_3\,, and \quad -\underline{K} = -\left(\frac{1}{\sqrt{3}}\underline{e}_1 + \frac{1}{\sqrt{3}}\underline{e}_2 + \frac{1}{\sqrt{3}}\underline{e}_3\right)$$

$$\underline{\underline{R}} = \underline{\underline{R}}(x_2, 90)\underline{\underline{R}}(x_3, 90) = \underline{\underline{R}}(\underline{K}, 120) = \underline{\underline{R}}(-\underline{K}, -120)$$

Infinitesimal Rotations, Angular Velocity, and Angular Acceleration:

Theorem-11: **For *general infinitesimal rotations*, sequential of the axes of rotation is not important.**
Let us consider the displacement of a body point in a rotating rigid body:



$\underline{r}^*$ is the vector $\underline{r}$ after $\underline{\underline{R}}$ Rotation, therefore:

$$\underline{r}^* = \underline{\underline{R}}\,\underline{r}$$ **, and**

$$\Delta\underline{r} = \underline{r}^* - \underline{r} = (\underline{\underline{R}} - \underline{\underline{I}})\underline{r} \quad \textbf{(4.10)}$$

If we let that x_1 and x_3 to be the axes of rotation, we have:

$$\underline{\underline{R}}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta_1 & -s\theta_1 \\ 0 & s\theta_1 & c\theta_1 \end{bmatrix}, \quad and \quad \underline{\underline{R}}_3 = \begin{bmatrix} c\theta_3 & -s\theta_3 & 0 \\ s\theta_3 & c\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underline{\underline{R}}_{13} = \underline{\underline{R}}_1 \underline{\underline{R}}_3 = \begin{bmatrix} c\theta_3 & -s\theta_3 & 0 \\ c\theta_1 s\theta_3 & c\theta_1 c\theta_3 & -s\theta_1 \\ s\theta_1 s\theta_3 & s\theta_1 c\theta_3 & c\theta_1 \end{bmatrix} \& \ \underline{\underline{R}}_{31} = \underline{\underline{R}}_3 \underline{\underline{R}}_1 = \begin{bmatrix} c\theta_3 & -s\theta_3 c\theta_1 & s\theta_3 s\theta_1 \\ s\theta_3 & c\theta_1 c\theta_3 & -s\theta_1 c\theta_3 \\ 0 & s\theta_1 & c\theta_1 \end{bmatrix}$$

$$\underline{\underline{R}}_{13} \neq \underline{\underline{R}}_{31} \qquad \textbf{(4.11)}$$

Now let: $\left\{ \begin{array}{l} \theta = \Delta\theta = O(\varepsilon) = very - small \Rightarrow \cos\theta \to 1, and \quad \sin\theta \to \Delta\theta \\ O(\varepsilon^2) = 0 \end{array} \right.$

Then:

$$\underline{\underline{R}}_{13} = \begin{bmatrix} 1 & -\Delta\theta_3 & 0 \\ \Delta\theta_3 & 1 & -\Delta\theta_1 \\ 0 & \Delta\theta_1 & 1 \end{bmatrix}, \quad and \quad \underline{\underline{R}}_{31} = \begin{bmatrix} 1 & -\Delta\theta_3 & 0 \\ \Delta\theta_3 & 1 & -\Delta\theta_1 \\ 0 & \Delta\theta_1 & 1 \end{bmatrix} \Rightarrow$$

$$\underline{\underline{R}}_{13} = \underline{\underline{R}}_{31}$$

Therefore, for *General Infinitesimal Rotations* we have:

$$\underline{\underline{R}} = \underline{\underline{R}}_1 \underline{\underline{R}}_2 \underline{\underline{R}}_3 = \underline{\underline{R}}_3 \underline{\underline{R}}_2 \underline{\underline{R}}_1 \equiv \begin{bmatrix} 1 & -\Delta\theta_3 & \Delta\theta_2 \\ \Delta\theta_3 & 1 & -\Delta\theta_1 \\ -\Delta\theta_2 & \Delta\theta_1 & 1 \end{bmatrix}, or \quad R_{jk} = \delta_{jk} - \gamma_{ijk}\Delta\theta_i \qquad \textbf{(4.12)}$$

The displacement vector $\Delta\underline{r}$ due to such combination of rotations will be:

$$\Delta\underline{r} = \underline{r}^* - \underline{r} = (\underline{\underline{R}} - \underline{\underline{I}})\underline{r} \qquad or \qquad \Delta x_i = (R_{ik} - \delta_{ik})x_k = -\gamma_{jik}\Delta\theta_j x_k = \gamma_{ijk}\Delta\theta_j x_k$$

In matrix form:

$$\Delta \underline{r} = \begin{bmatrix} 0 & -\Delta\theta_3 & \Delta\theta_2 \\ \Delta\theta_3 & 0 & -\Delta\theta_1 \\ -\Delta\theta_2 & \Delta\theta_1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \textbf{(4.13)}$$

$$\Delta x_i = \gamma_{ijk} \Delta\theta_j x_k \qquad \Rightarrow \Delta \underline{r} = \Delta \underline{\theta} \times \underline{r}$$

(from the definition of cross product)

Since $\underline{r}$ is a vector of constant magnitude in the rigid body, we have:

$$\lim_{\Delta t \to 0} \frac{\Delta \underline{r}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta \underline{\theta}}{\Delta t} \times \underline{r} \qquad \Rightarrow \qquad \dot{\underline{r}} = \underline{\omega} \times \underline{r} \qquad where;$$

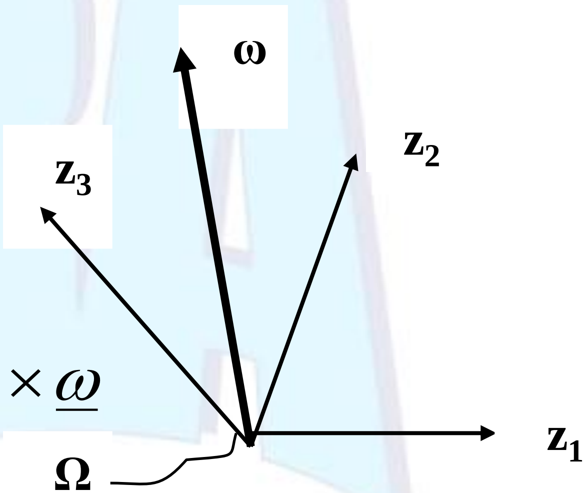
$$\underline{\omega} = \lim_{\Delta t \to 0} \frac{\Delta \underline{\theta}}{\Delta t} = \omega_i \underline{e}_i \qquad (\underline{Angular-Velocity}) \qquad \textbf{(4.14)}$$

$$\dot{\underline{r}} = \lim_{\Delta t \to 0} \frac{\Delta \underline{r}}{\Delta t}$$

Angular Velocity Vector:

$\underline{\omega} = \omega_i \underline{e}_i$ = (sum of the rotation rates about various axes).

If the angular velocity vector " $\underline{\omega}$ " is defined (expressed) in a set of *moving coordinates* {z_i} having an angular velocity "$\underline{\Omega}$", we may apply the *Jaumann* rate of a vector to compute the angular acceleration vector.



Angular Acceleration Vector:

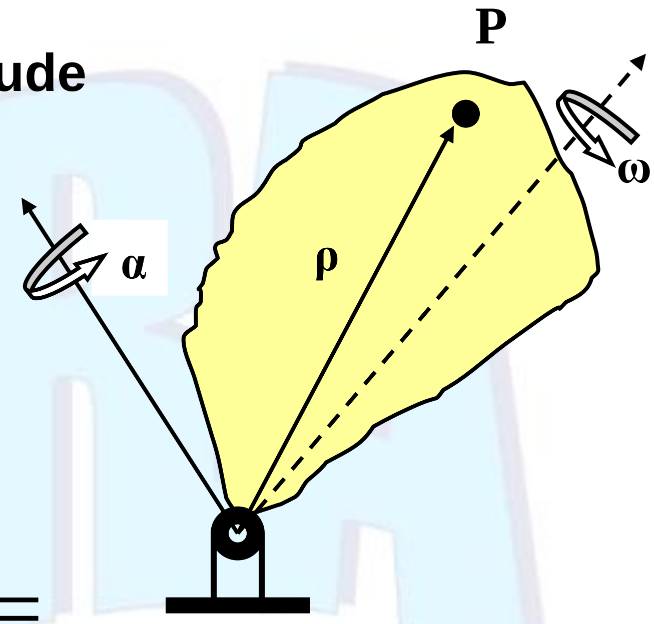
$$\underline{\alpha} = \dot{\omega}_i \underline{e}_i + \omega_i \dot{\underline{e}}_i = \dot{\omega}_i \underline{e}_i + \omega_i (\underline{\Omega} \times \underline{e}_i) = \dot{\omega}_i \underline{e}_i + \underline{\Omega} \times \underline{\omega} \quad (4.15)$$

Note: Even if the rotation rates are constant, there will be an angular acceleration whenever any of the axes do not have a fixed orientation.

Velocity and Acceleration Field in a Rotating Rigid Body:

Consider the rotating rigid body shown:

***Given*: $\underline{\omega}$ and $\underline{\alpha}$ as shown, and**
$\underline{\rho}$: vector of constant magnitude fixed in the rigid body.



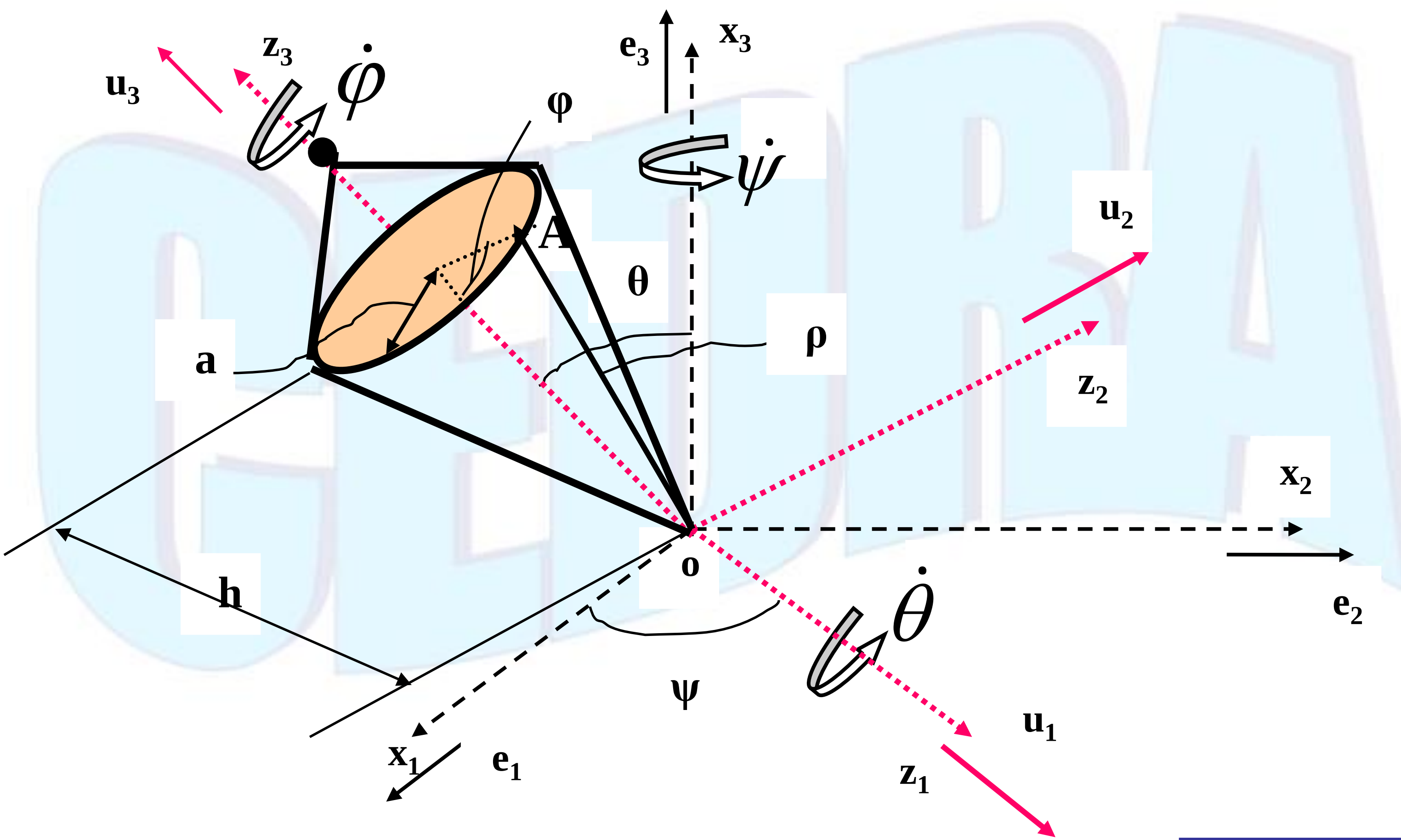
***We have*:**

$$\underline{v}_P = \underline{\dot{\rho}} = \underline{\omega} \times \underline{\rho} \qquad (4.16)$$

$$\underline{a}_P = \underline{\dot{v}}_P = \underline{\dot{\omega}} \times \underline{\rho} + \underline{\omega} \times \underline{\dot{\rho}} =$$

$$\underline{a}_P = \underline{\alpha} \times \underline{\rho} + \underline{\omega} \times (\underline{\omega} \times \underline{\rho}) \qquad (4.17)$$

Ex: A top *spins* at a constant speed " $\dot{\varphi}$ " at a fixed spot "O". Meanwhile, it *precesses* about the vertical axis at a speed " $\dot{\psi}$ ", and its altitude *nutates* at a speed " $\dot{\theta}$ " as shown. Determine the velocity and acceleration of a point "A" on its upper rim?



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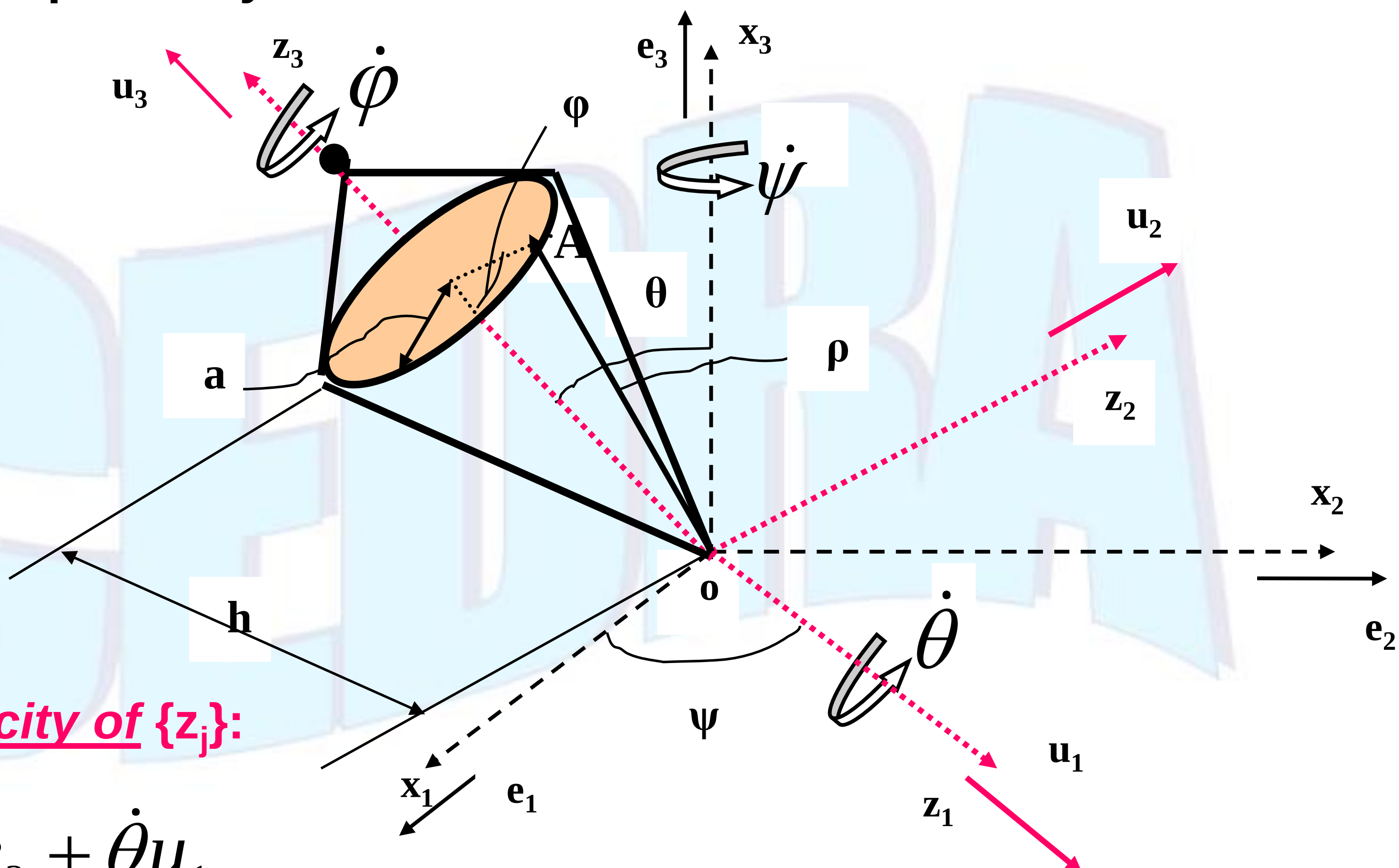
Step-1: Angular Motion Analysis

$\{x_i\}$: fixed spatial coordinate.

$\{z_j\}$: rotating coordinate in which the top spins only.



Angular Velocity of $\{z_j\}$:

$$\underline{\Omega} = \dot{\psi}\underline{e}_3 + \dot{\theta}\underline{u}_1$$

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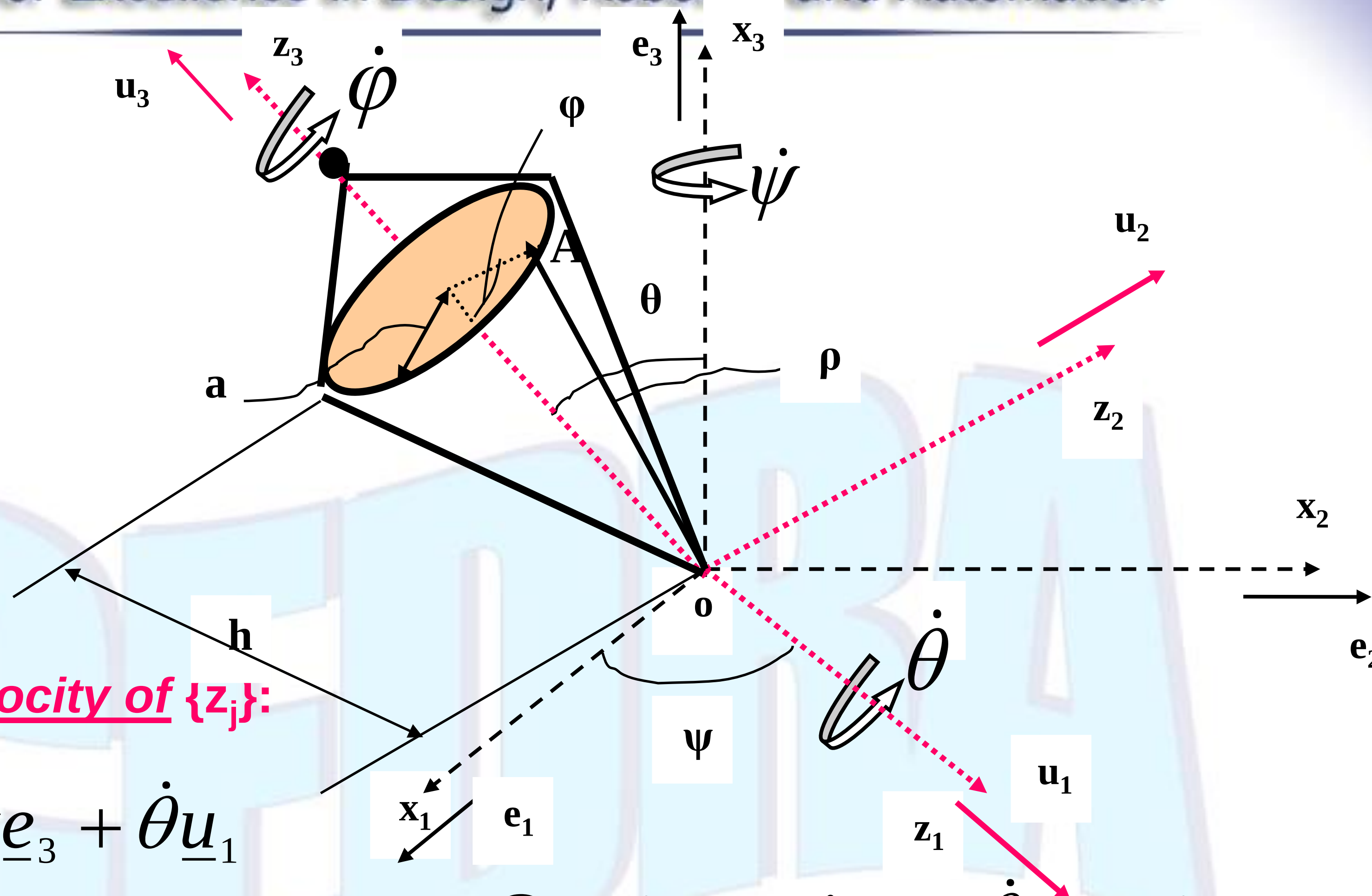


Angular Velocity of {z_j}:

$$\underline{\Omega} = \dot{\psi}\underline{e}_3 + \dot{\theta}\underline{u}_1$$

Angular Velocity of the top: $\underline{\omega} = \underline{\Omega} + \dot{\varphi}\underline{u}_3 = \dot{\psi}\underline{e}_3 + \dot{\theta}\underline{u}_1 + \dot{\varphi}\underline{u}_3$

***In terms of the rotating coordinate* {z_j}:**

$$\underline{\Omega} = \dot{\theta}\underline{u}_1 + \dot{\psi}\sin\theta\underline{u}_2 + \dot{\psi}\cos\theta\underline{u}_3$$

$$\underline{\omega} = \dot{\theta}\underline{u}_1 + \dot{\psi}\sin\theta\underline{u}_2 + (\dot{\varphi} + \dot{\psi}\cos\theta)\underline{u}_3$$

Angular Acceleration of the top: **Note that $\underline{\omega}$ is expressed in a set of moving coordinate $\{z_j\}$, to find $\underline{\alpha}$ use the *Jaumann* rate of a vector as:**

$$\underline{\alpha} = \dot{\omega}_i \underline{e}_i + \underline{\Omega} \times \underline{\omega} = \ddot{\theta}\underline{u}_1 + (\ddot{\psi}\sin\theta + \dot{\psi}\dot{\theta}\cos\theta)\underline{u}_2 + (\ddot{\psi}\cos\theta - \dot{\psi}\dot{\theta}\sin\theta)\underline{u}_3 +$$

$$(\dot{\theta}\underline{u}_1 + \dot{\psi}\sin\theta\underline{u}_2 + \dot{\psi}\cos\theta\underline{u}_3) \times [\dot{\theta}\underline{u}_1 + \dot{\psi}\sin\theta\underline{u}_2 + (\dot{\varphi} + \dot{\psi}\cos\theta)\underline{u}_3] =$$

$$\underline{\alpha} = (\ddot{\theta} + \dot{\varphi}\dot{\psi}\sin\theta)\underline{u}_1 + (\ddot{\psi}\sin\theta + \dot{\psi}\dot{\theta}\cos\theta - \dot{\varphi}\dot{\theta})\underline{u}_2 + (\ddot{\psi}\cos\theta - \dot{\psi}\dot{\theta}\sin\theta)\underline{u}_3$$

Step-2: Absolute Motion Analysis

***Position Vector of A in* $\{z_j\}$:**

$$\underline{\rho} = a\cos\varphi\underline{u}_1 + a\sin\varphi\underline{u}_2 + h\underline{u}_3$$

***Velocity Vector*:**

$$\underline{v}_A = \underline{\omega} \times \underline{\rho}$$

***Acceleration Vector*:**

$$\underline{a}_A = \underline{\alpha} \times \underline{\rho} + \underline{\omega} \times (\underline{\omega} \times \underline{\rho})$$

For example if $\dot{\varphi}=1000$ **rpm=constant** $\Rightarrow \varphi = 1000(\frac{2\pi}{60})t$ **rad**, $\theta = 0.2\sin(100\pi t)$ **rad, and** $\psi = 0.5\sin(50\pi t)$ **rad, then for a specific time (i.e. 2 ms) one can covert the results presented in rotating coordinate-{z_j}, to the space-fixed coordinate-{x_i}, using the rotation transformation "** $\underline{\underline{R}} = \underline{\underline{R}}_{\psi}\,\underline{\underline{R}}_{\theta}\,\underline{\underline{R}}_{\varphi}$ **" between the two coordinates as:**

$$\{\underline{\rho}\}_x = \underline{\underline{R}}\{\underline{\rho}\}_z, \quad \{\underline{v}_A\}_x = \underline{\underline{R}}\{\underline{v}_A\}_z, \quad \{\underline{a}_A\}_x = \underline{\underline{R}}\{\underline{a}_A\}_z, \quad \{\underline{\omega}\}_x = \underline{\underline{R}}\{\underline{\omega}\}_z, \quad \{\underline{\alpha}\}_x = \underline{\underline{R}}\{\underline{\alpha}\}_z$$

General Motion of a Rigid Body (Translation & Rotation):

Chasle's Theorem(12): **The general motion of a rigid body can be described by a combination of motion of some convenient reference body point and an *Eulerian* rotation about that point.**

Note: A rigid body in space possesses *Six-Degrees-of-Freedom*:

3-DOF: for the *position* of the reference point (rigid body), and

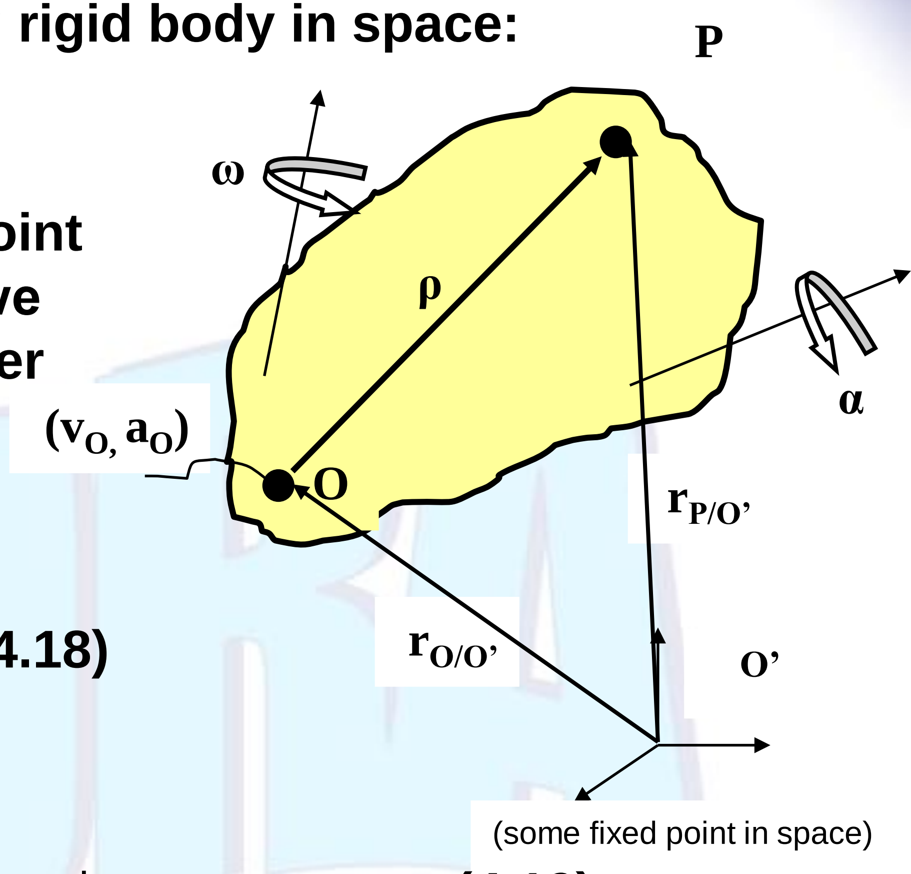
3-DOF: for the *orientation* of the rigid body (i.e. Euler's Angles).

Let us consider the motion of a moving rigid body in space:

Let us consider the motion of a moving rigid body in space:

$\underline{\omega} = \omega_i \underline{e}_i$, **and** $\underline{\alpha} = \alpha_i \underline{e}_i$

Suppose that the motion of a body point "O" in the rigid body in known, and we wish to compute the motion of another body point "P"?



Position Vector:

$$\underline{r}_{P/O'} = \underline{r}_{O/O'} + \underline{\rho} \qquad \textbf{(4.18)}$$

Velocity Vector:

$$\underline{v}_P = \frac{d\underline{r}_{P/O'}}{dt} = \dot{\underline{r}}_{O/O'} + \dot{\underline{\rho}} = \underline{v}_O + \underline{\omega} \times \underline{\rho} \qquad \textbf{(4.19)}$$

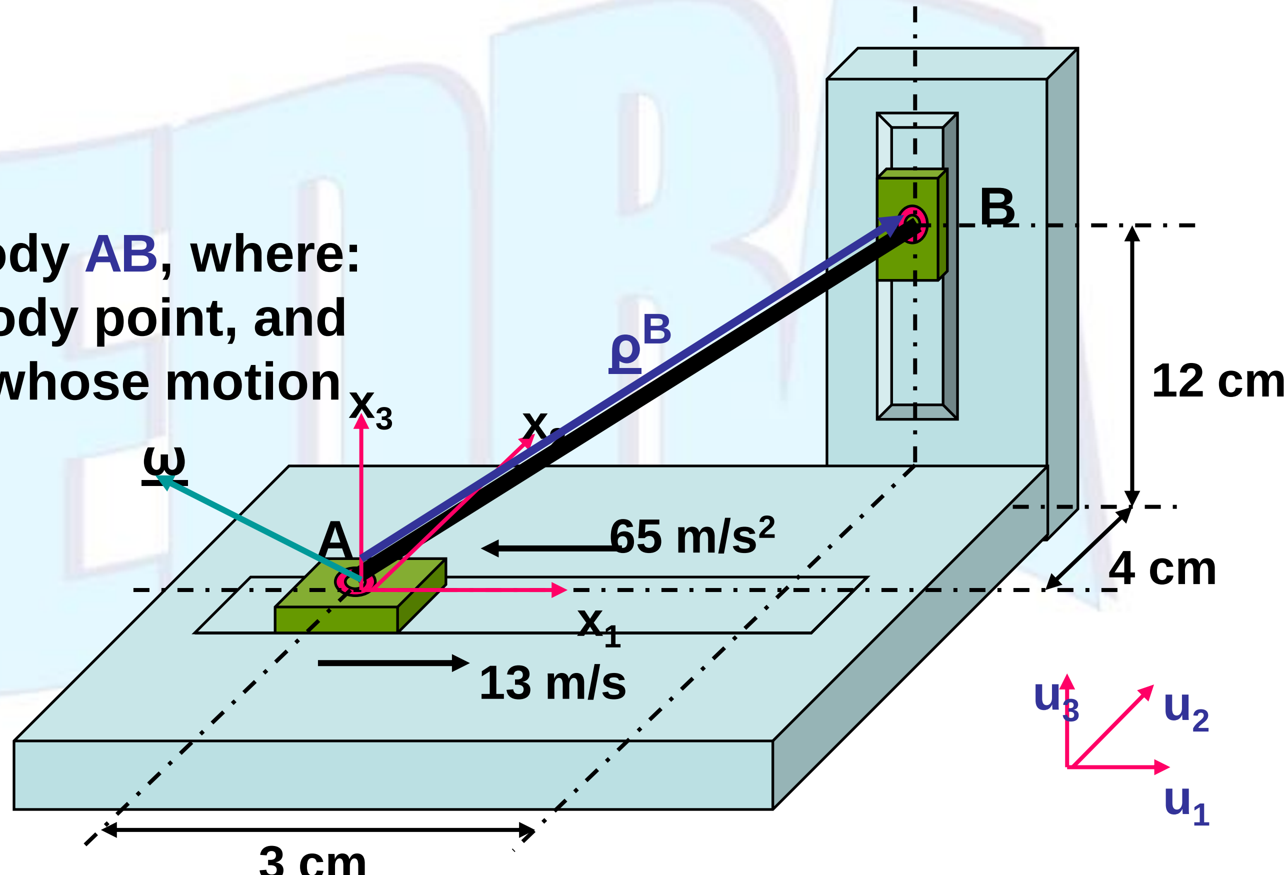
Acceleration Vector:

$$\underline{a}_P = \underline{a}_O + \underline{\alpha} \times \underline{\rho} + \underline{\omega} \times (\underline{\omega} \times \underline{\rho}) \qquad \textbf{(4.20)}$$

Ex: **Given the velocity and acceleration of the block "A" in a double slide mechanism shown. Determine the velocity and acceleration of the block "B"? The hinges at A and B are both of the ball-and -socket type.**

Given: $\underline{v}_A$ and $\underline{a}_A$,
Find: $\underline{v}_B$ and $\underline{a}_B$ = ?

Solution:
Consider the rigid body AB, where:
A: is the reference body point, and
B: is the body point whose motion is to be analyzed.



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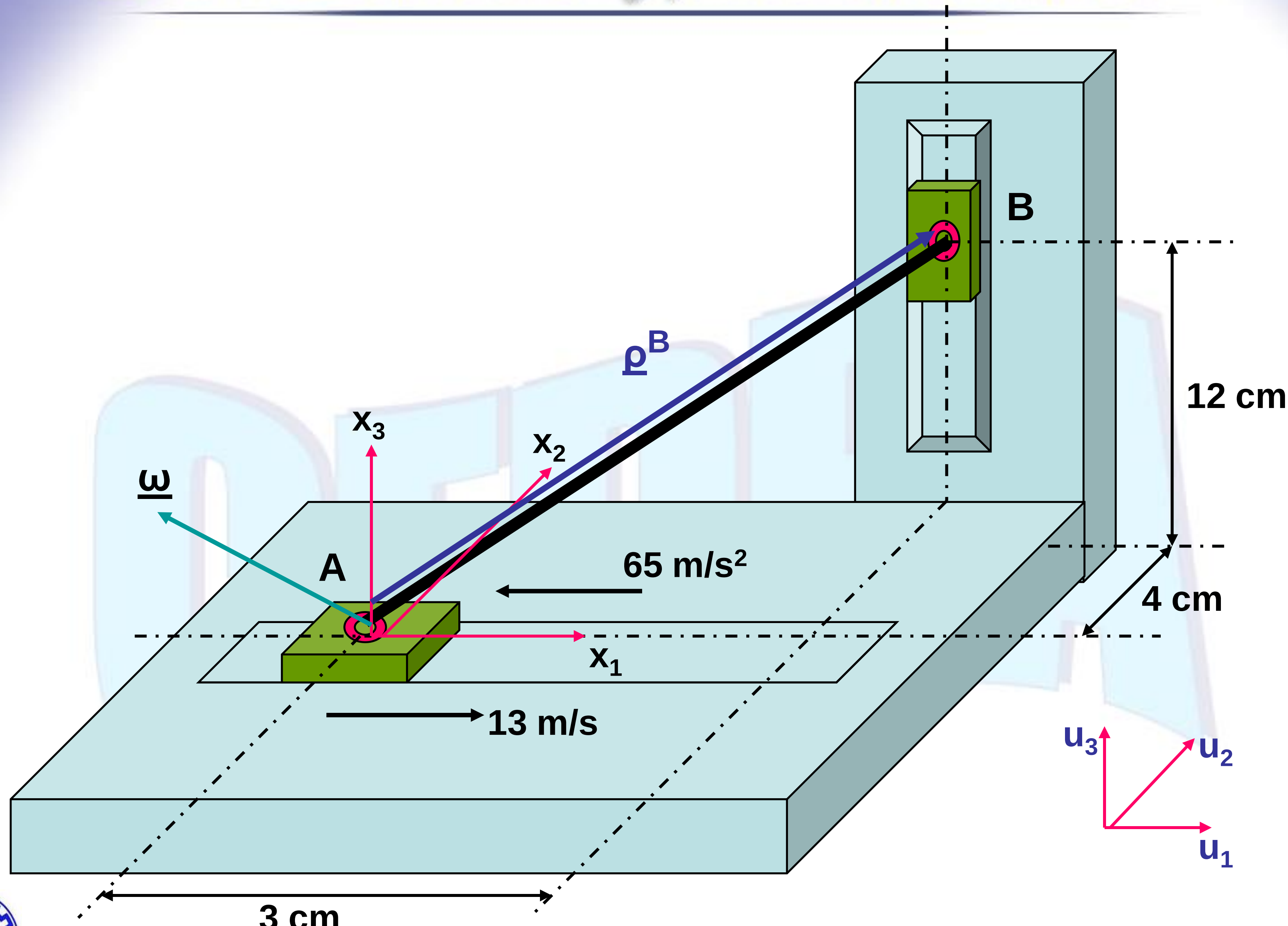

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Step-1: ***Motion*** **(pos., vel., acc.) of point "B";**

$$\underline{\rho}^{B} = 3\underline{u}_1 + 4\underline{u}_2 + 12\underline{u}_3 \quad \textbf{cm}$$

$$\underline{v}_B = v_B \underline{u}_3 , \quad and \quad \underline{a}_B = a_B \underline{u}_3$$

Step-2: ***Motion*** **of reference point "A";**

$$\underline{v}_A = 1300\underline{u}_1 \, cm/s, \quad and \quad \underline{a}_A = -6500\underline{u}_1 \, cm/s^2$$

Step-3: **The *angular motions* of AB is unknown. Since the axial component of the angular motions of AB do not contribute to the motion of point "B", we *impose the following conditions*;**

$$\left\{ \begin{aligned} \underline{\omega} \cdot \underline{e}_{B/A} &= \underline{\omega} \cdot \frac{\underline{\rho}^{B}}{\left|\underline{\rho}^{B}\right|} = 0 \\ \underline{\alpha} \cdot \underline{e}_{B/A} &= \underline{\alpha} \cdot \frac{\underline{\rho}^{B}}{\left|\underline{\rho}^{B}\right|} = 0 \end{aligned} \right\} \Rightarrow \left\{ \begin{aligned} \underline{\omega} \cdot \underline{\rho}^{B} &= 3\omega_1 + 4\omega_2 + 12\omega_3 = 0 \qquad (a) \\ \underline{\alpha} \cdot \underline{\rho}^{B} &= 3\alpha_1 + 4\alpha_2 + 12\alpha_3 = 0 \qquad (b) \end{aligned} \right\}$$

Step-4: ***Velocity Analysis***;

$$\underline{v}_B = \underline{v}_A + \underline{\omega} \times \underline{\rho}^B$$

$$v_B \underline{u}_3 = 1300\underline{u}_1 + (\omega_1 \underline{u}_1 + \underline{\omega}_2 \underline{u}_2 + \underline{\omega}_3 \underline{u}_3) \times (3\underline{u}_1 + 4\underline{u}_2 + 12\underline{u}_3)$$

$$\left\{\begin{array}{lll} \underline{u}_1: & 0 = 1300 + 12\omega_2 - 4\omega_3 & (c) \\ \underline{u}_2: & 0 = 3\omega_3 - 12\omega_1 & (d) \\ \underline{u}_3: & v_B = 4\omega_1 - 3\omega_2 & (e) \end{array}\right\}$$

From expressions (a, c, d, e), we have:

$$\underline{v}_B = 325\underline{u}_3 \, cm/s, \quad and \quad \underline{\omega} = (100\underline{u}_1 - 1275\underline{u}_2 + 400\underline{u}_3)/13 \quad rad/s$$

Step-5: ***Acceleration Analysis***;

$$\underline{a}_B = \underline{a}_A + \underline{\alpha} \times \underline{\rho}^B + \underline{\omega} \times (\underline{\omega} \times \underline{\rho}^B) \quad \Rightarrow$$

$$\begin{cases} \underline{u}_1 : & 0 = -6500 + 12\alpha_2 - 4\alpha_3 - 31875 & (f) \\ \underline{u}_2 : & 0 = 3\alpha_3 - 12\alpha_1 - 42500 & (g) \\ \underline{u}_3 : & a_B = 4\alpha_1 - 3\alpha_2 - 127500 & (h) \end{cases}$$

From expressions (b, f, g, h) , we have:

$$\underline{a}_B = -151260\underline{u}_3 \, cm/s^2, \quad and \quad \underline{\alpha} = -3580\underline{u}_1 + 3146.6\underline{u}_2 - 153.8\underline{u}_3 \, rad/s^2$$

Also: $x_1^2 + x_2^2 + x_3^2 = \ell^2 = constant$

$$\frac{d}{dt}(......) = 2\dot{x}_1 x_1 + 2\underbrace{\cancel{\dot{x}_2}}_{0} x_2 + 2\dot{x}_3 x_3 = 0 \quad \Rightarrow \quad \dot{x}_3 = ?$$

$$\frac{d}{dt}(......) = \ddot{x}_1 x_1 + \dot{x}_1^2 + \ddot{x}_3 x_3 + \dot{x}_3^2 = 0 \quad \Rightarrow \quad \ddot{x}_3 = ?$$

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