بسم الله الرحمن الرحیم

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RIGID BODY KINEMATICS

***Purpose*:**

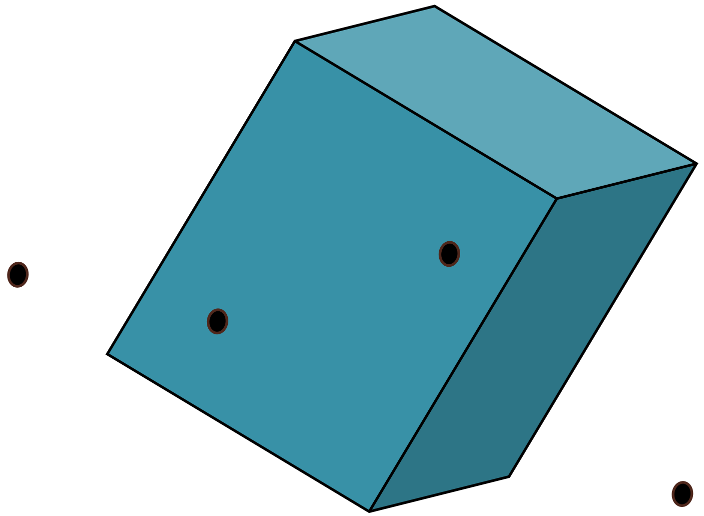
- **Analytical Description of Rigid Body Motion.**
- **Matrix Transforms to Represent Rigid Body Motion.**
- **Reinforcement of Elementary Kinematical Equations.**

***Topics*:**

- **Translation of Rigid Bodies.**
- **Rotation of Rigid Bodies.**
- **General Motion of Rigid Bodies (i.e. Robot Kinematics)**
- **Coordinate Transformations**

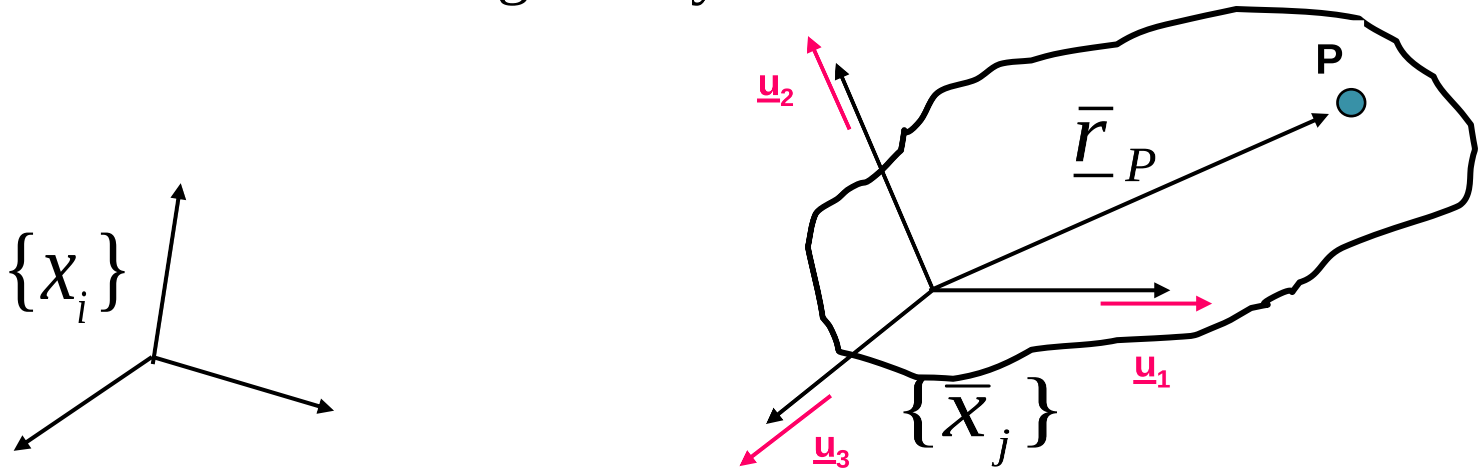
***Rigid Body*:** **The simplest form of a continuum, is an aggregate of particles of which the distance between any pair remains constant throughout the dynamic process.**

***Body Point*:** **A point which is fixed in a rigid body or its imaginary extension throughout the motion.**



Spatial Coordinate System: A coordinate system , $\{x_i\}$ that is fixed in space in which the rigid body moves.

Rigid Body Coordinate System: A coordinate system, $\{\bar{x}_j\}$ that is fixed to the rigid body and moves with it.



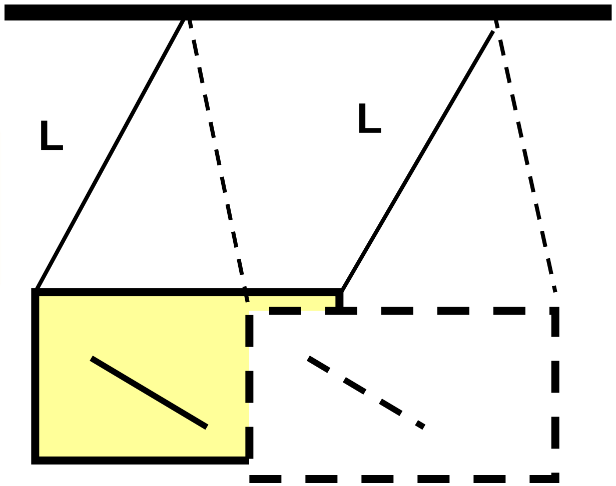
Note: **The position of a body point in its body coordinate is invariant.**

$\underline{\bar{r}}_P = \bar{x}_j \underline{u}_j$ = (***An Invariant Set***), **therefore:** **(4.1)**

$\{\dot{\bar{x}}_j\} = \{0\}$ = (***Velocity of body points relative to the body coordinate is zero***)

Translation: **When the line segment connecting any pair of the body points in a moving rigid body maintains its orientation during the motion, we say that the rigid body is in *pure translation*.**

Ex: *Rigid Body in Translation*,
But particles may be in circular motion.



Theorem-7: **In a translating rigid body, the displacement of all body points are the same.**

Theorem-8: **All body points in a translating rigid body have simultaneous equal velocities and equal accelerations.**

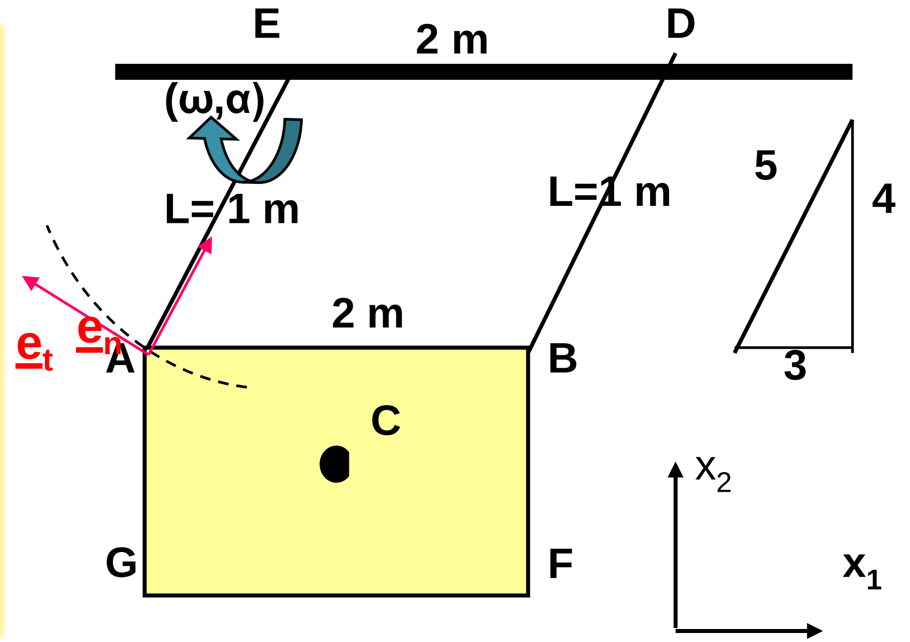
Result: **In kinematical analysis, once you are sure that a rigid body has invariant orientation, any convenient body point may be selected to compute its motion.**

Ex: **Consider the rectangular block shown:**

ABED : Parallelogram

$$\underline{\omega}_{Block} = \underline{\alpha}_{Block} = 0$$

(block stays always horizontal, therefore it is in pure translation)



Given: **ω = 20 rad/s, α = 300 rad/s², Find $\underline{a}_c$ = ?**

Solution: $\underline{a}_C = \underline{a}_A = R\alpha\underline{e}_t + R\omega^2\underline{e}_n = 300(\frac{-4}{5}\underline{e}_1 + \frac{3}{5}\underline{e}_2) + 400(\frac{3}{5}\underline{e}_1 + \frac{4}{5}\underline{e}_2) = 500\underline{e}_2 m/s^2$

Rotation: **When a moving rigid body has a fixed (or momentarily fixed) body point in space, the rigid body is said to be in *rotation*.**

Theorem-9: **In a rotating rigid body, when more than one fixed body points are present, these points must lie on a common straight line called the *fixed (instantaneous) axis of rotation* of rigid body.**

We previously showed that: *Rotation is an Orthogonal Transform*. If $\{X_i\}$ is a spatial coordinate system, and $\{\overline{X}_j\}$ is the rigid body coordinate system, we have:

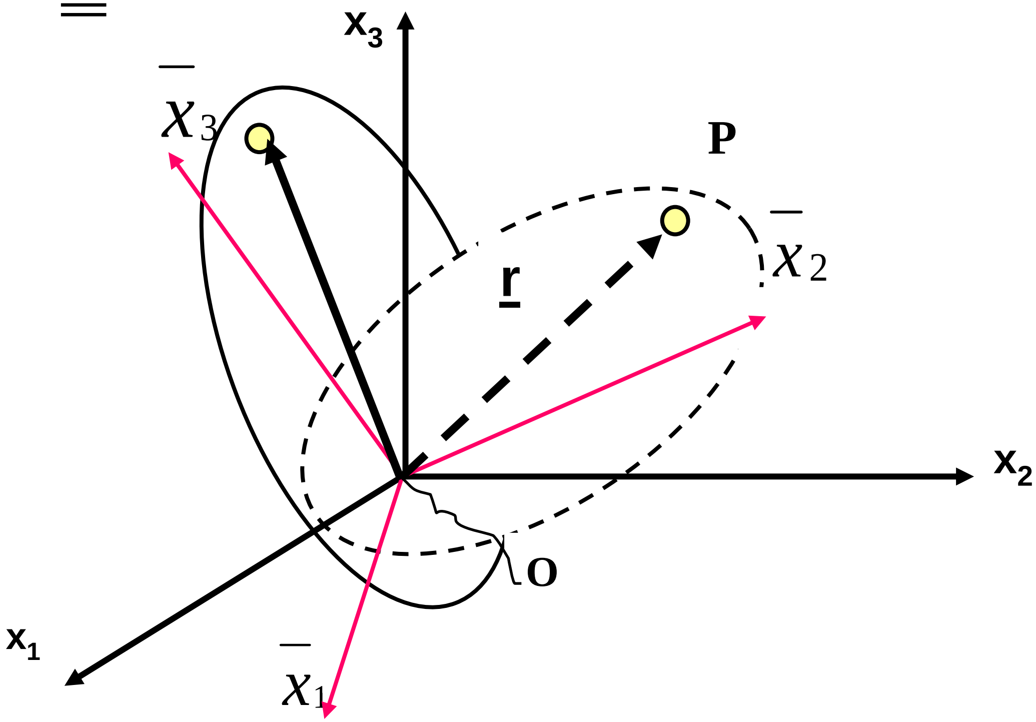
$$\{x_i^P\} = \underline{\underline{R}}\{\overline{x}_j^P\} \qquad \textbf{(4.2)}$$

$$\{x_i^P\} = \underline{\underline{R}}\{\bar{x}_j^P\} \quad (4.2)$$

where: $\{x_i^P\}$**: Spatial position of "P" after rotation.**

$\{\bar{x}_j^P\}$**: Original position of "P".**

$\underline{\underline{R}}$ **: Rotation Matrix.**



Note: For a rigid body, the distance between two body points "$\overline{OP}$" is *constant*.

$$\overline{OP} = |\underline{r}| = \underline{r} \cdot \underline{r} = \{x_i^P\}^t \{x_i^P\} = [x_1 \quad x_2 \quad x_3] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

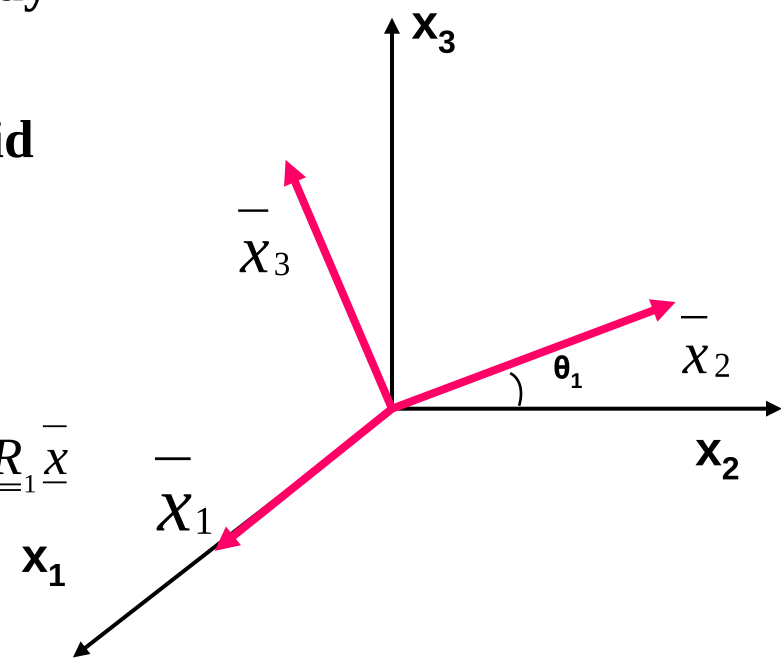
$$\{x_i^P\}^t = \{\bar{x}_j^P\}^t \underline{\underline{R}}^t$$

$$\underline{r} \cdot \underline{r} = \{\bar{x}_j^P\}^t \underline{\underline{R}}^t \underline{\underline{R}} \{\bar{x}_j^P\} = \{\bar{x}_j^P\}^t \{\bar{x}_j^P\} = \{x_i^P\}^t \{x_i^P\}$$ **=** ***constant***

Simple Rotation: **Rotation of a rigid body about a general fixed axes in space.**

Elementary Rotation: **Rotation of a rigid body about one of the coordinate axes.**

$$\underline{\underline{R}}_1(x_1, \theta_1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{bmatrix} \Rightarrow \underline{x} = \underline{\underline{R}}_1 \underline{\bar{x}}$$



***Euler's Theorem(10)*:**

Any change of orientation (about an arbitrary axis) for a rigid body with a fixed body point can be accomplished through a *simple rotation* [Eq. (4.2)]. Then,the rigid body rotation can be resolved into three *elementary rotations*, where the angles of these rotationsare called the *Euler's Angles*.

***Finite (Spatial) Rotation*:**

A spatial rotation features rotation about two or more nonparallel coordinate axes. Note that finite rotation is order dependent, and *does not satisfy the Commutative Law*.

***Two situations commonly arise in sequential rotations*:**

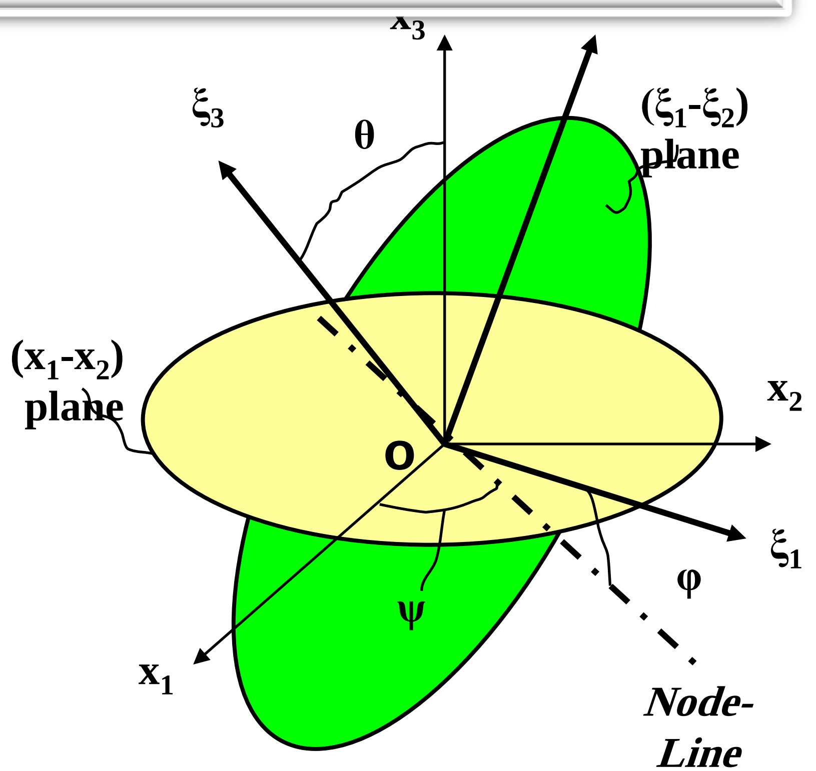
1. **Body Fixed Rotations (*Rotations about New-axis*)**
2. **Space Fixed Rotations (*Rotations about Original/Old-axis*)**

1. Body Fixed Rotations (Rotations about New-axis)

Consider the following coordinates:

$\{x_i\}$: Spatial Coordinates

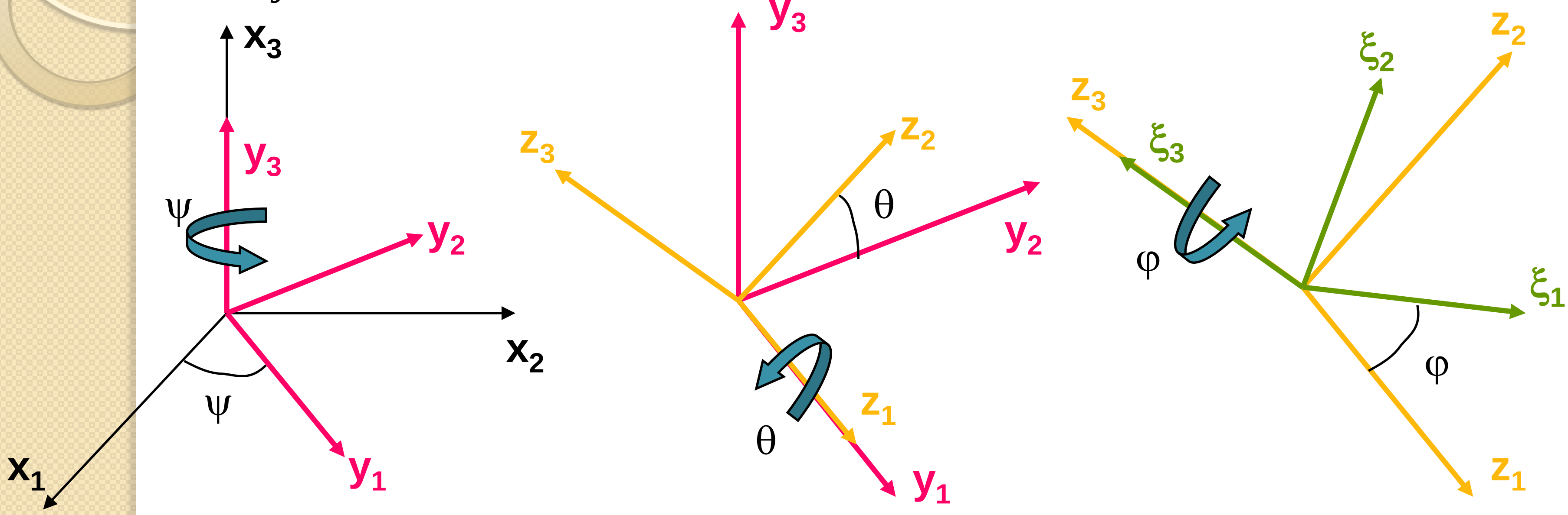
$\{\xi_j\}$: Body Coordinates



$$\{x_i\} = \underline{\underline{R}}\{\xi_j\}$$

$\underline{\underline{R}}$: **Simple Rotation about a general axis in space** (can be resolved into 3-elementary rotations).

Let us start from $\{x_i\}$ and with a sequence of three rotations obtain $\{\xi_j\}$:



$$\underline{x} = \underline{\underline{R}}_{\psi}\, \underline{y} \qquad \underline{y} = \underline{\underline{R}}_{\theta}\, \underline{z} \qquad \underline{z} = \underline{\underline{R}}_{\varphi}\, \underline{\xi}$$

where:

$$\underline{\underline{R}}(x_3,\psi) = \begin{bmatrix} c\psi & -s\psi & 0 \\ s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \underline{\underline{R}}(y_1,\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta & -s\theta \\ 0 & s\theta & c\theta \end{bmatrix}, \quad \underline{\underline{R}}(z_3,\varphi) = \begin{bmatrix} c\varphi & -s\varphi & 0 \\ s\varphi & c\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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Therefore:

$$\underline{x} = \underline{\underline{R}}_{\psi}\,\underline{y} = \underline{\underline{R}}_{\psi}\,\underline{\underline{R}}_{\theta}\,\underline{z} = \underline{\underline{R}}_{\psi}\,\underline{\underline{R}}_{\theta}\,\underline{\underline{R}}_{\varphi}\,\underline{\xi} \quad \Rightarrow \quad \underline{x} = \underline{\underline{R}}\,\underline{\xi} \qquad \textbf{(4.3)}$$

where:

$$\underline{\underline{R}} = {}^{x}_{\xi}\underline{\underline{R}} = \underline{\underline{R}}_{\psi}\,\underline{\underline{R}}_{\theta}\,\underline{\underline{R}}_{\varphi} = \begin{bmatrix} c\psi c\varphi - s\psi c\theta s\varphi & -c\psi s\varphi - s\psi c\theta c\varphi & s\psi s\theta \\ s\psi c\varphi + c\psi c\theta s\varphi & -s\psi s\varphi + c\psi c\theta c\varphi & -c\psi s\theta \\ s\theta s\varphi & s\theta c\varphi & c\theta \end{bmatrix} \qquad \textbf{(4.4)}$$

or:

$$\underline{x} = {}^{x}_{\xi}\underline{\underline{R}}\,\underline{\xi} = (\underline{\underline{R}}_{1}\,\underline{\underline{R}}_{2}\,\underline{\underline{R}}_{3})\,\underline{\xi} \qquad \textbf{(4.5)}$$

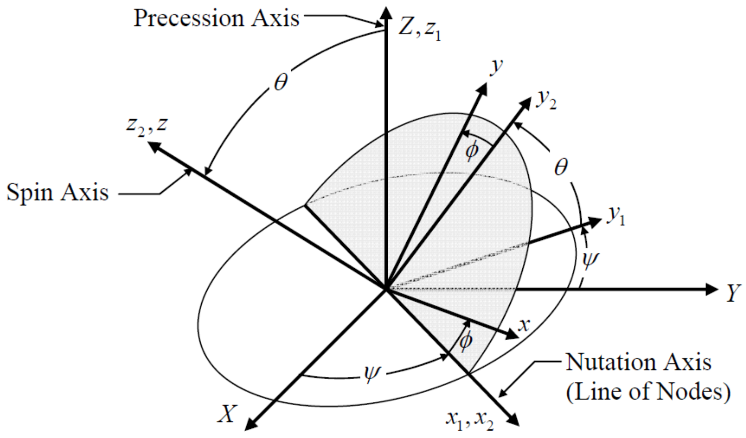
$$\underline{\underline{R}}_{1} = \underline{\underline{R}}_{\psi}, \qquad \underline{\underline{R}}_{2} = \underline{\underline{R}}_{\theta}, \qquad \underline{\underline{R}}_{3} = \underline{\underline{R}}_{\varphi}$$

Therefore, for "n" rotations:

$$\underline{\underline{R}} = \underline{\underline{R}}_{1}\,\underline{\underline{R}}_{2}\,\underline{\underline{R}}_{3}\ldots\underline{\underline{R}}_{n} \qquad \textbf{(4.6)}$$

= (***Post-Multiplication***)

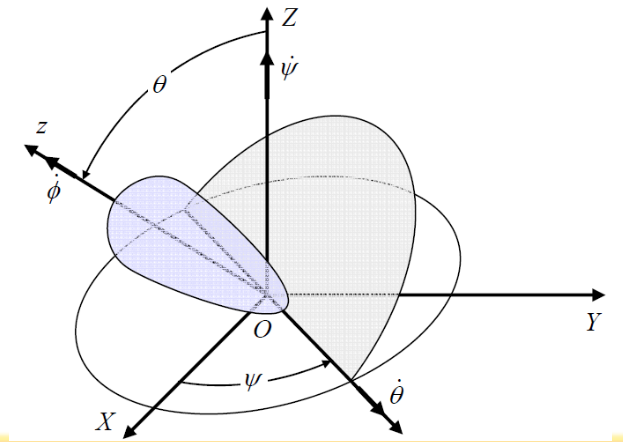
It is customary to refer to these angles as (***i.e. in Spinning Top or a Gyroscope***):



ψ : ***Angle of Precession***

θ : ***Angle of Nutation***

φ : ***Spin Angle***



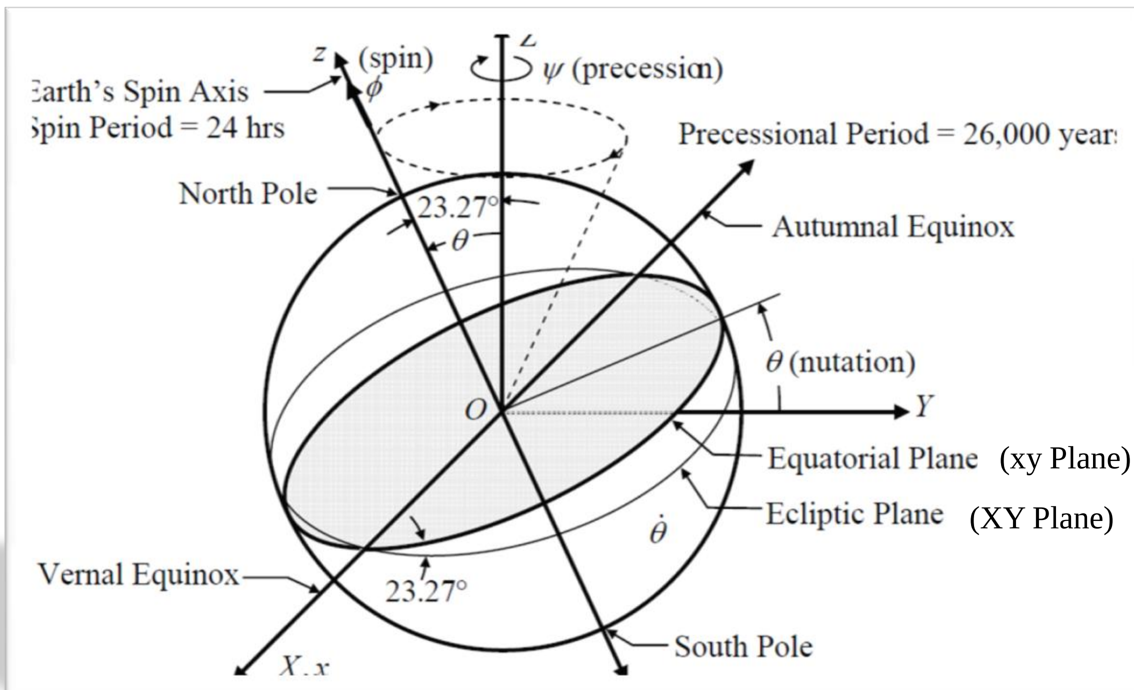
As a follow up to Euler's Theorem, angles "ψ, θ, φ" are called the *Euler's Angles*.

Ex: Describing the Earth's Motions in Space. In addition to its movement about the Sun in an Ecliptic orbit in the Ecliptic plane, the planet earth experiences Precession, Nutation, and Spin.

Spin: Daily rotation in 24 hours

Nutation: Inclination of the earth's spin axis with respect to the normal to the ecliptic plane (23.27°).

Precession: Earth's Spin-axis precesses along the surface of a hypothetical cone with apex angle of 23.27° , approximately once every 26000 years.



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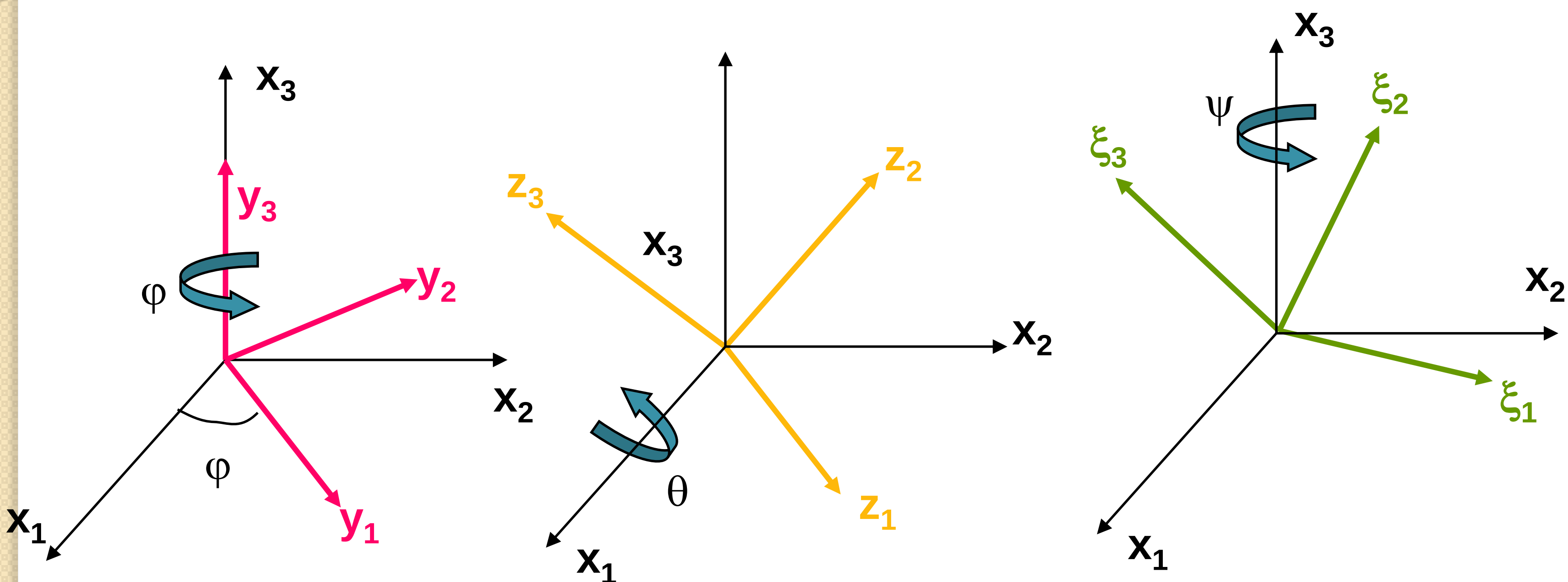
2. Space Fixed Rotations (Rotations about Original/Old-axis)Let us repeat a similar example, but this time start from and with a sequence of three rotations about the original/old axis obtain $\{\xi_j\}$**:**



It can be shown that (*proof in the Ginsberg's book*):

$$\underline{x} = {}^{x}_{\xi}\underline{\underline{R}}\,\underline{\xi} = \underline{\underline{R}}_{\psi}\,\underline{\underline{R}}_{\theta}\,\underline{\underline{R}}_{\varphi}\,\underline{\xi} = \underline{\underline{R}}_{3}\,\underline{\underline{R}}_{2}\,\underline{\underline{R}}_{1}\,\underline{\xi}$$ **; where:** **(4.7)**

$$\underline{\underline{R}}_1 = \underline{\underline{R}}_{\varphi} = \underline{\underline{R}}(x_3, \varphi); \qquad \underline{\underline{R}}_2 = \underline{\underline{R}}_{\theta} = \underline{\underline{R}}(x_1, \theta); \qquad \underline{\underline{R}}_3 = \underline{\underline{R}}_{\psi} = \underline{\underline{R}}(x_3, \psi)$$

Therefore, for "n" rotations:

$$\underline{\underline{R}} = \underline{\underline{R}}_n \ldots \underline{\underline{R}}_3\,\underline{\underline{R}}_2\,\underline{\underline{R}}_1$$ = **(*Pre-multiplication*)** **(4.8)**

(Note that my conventions for Pre & Post Multiplications are opposite to that of the Ginsberg's book, since I am defining $^{x}_{\xi}\underline{\underline{R}}$ *, but he is defining* $^{\xi}_{x}\underline{\underline{R}} \equiv {}^{x}_{\xi}\underline{\underline{R}}^{t} \equiv {}^{x}_{\xi}\underline{\underline{R}}^{-1}$ *).*

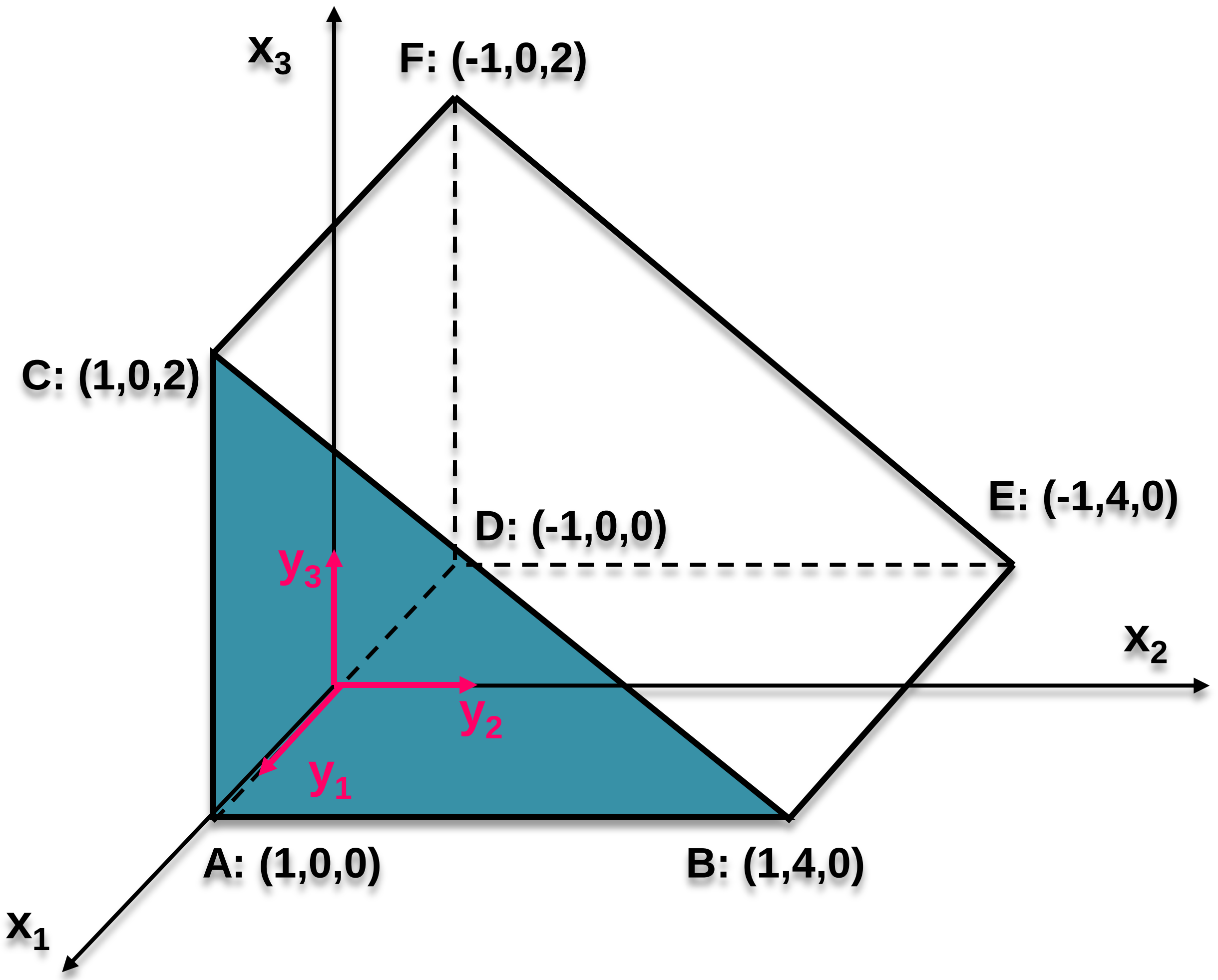
Special Case: A sequence of rotations described by $\underline{\underline{R}}_1$ and $\underline{\underline{R}}_2$ about Body Fixed (New) axes, followed by $\underline{\underline{R}}_3$ about a Space Fixed (Original) axis, and then $\underline{\underline{R}}_4$ about a Body Fixed (New) axis, would lead to:

$$\underline{\underline{R}} = \underline{\underline{R}}_3 \underline{\underline{R}}_1 \underline{\underline{R}}_2 \underline{\underline{R}}_4 \qquad and \qquad \underline{\underline{R}}^t = \underline{\underline{R}}_4^t \underline{\underline{R}}_2^t \underline{\underline{R}}_1^t \underline{\underline{R}}_3^t \qquad \textbf{(4.9)}$$

Note: The final orientation of a coordinate system depends on the sequence in which rotations occur, as well as the magnitude of the individual rotations and the orientation of their respective axes.

Important: Finite spatial rotations cannot be represented as vectors, because vector addition is independent of the order of addition.

Ex: Consider the object shown. First rotate it about the x_3-axis by 90^o, and then rotate it about the x_2-axis by 90^o. Determine the new orientation of the object?



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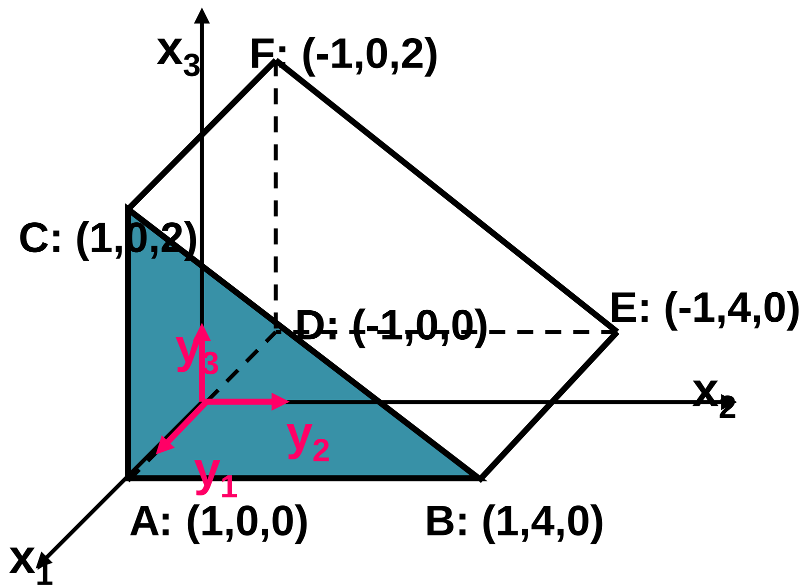
Solution: {$\underline{y}$} and {$\underline{x}$} originally coincide. Since all rotations are about the **Original (fixed) {x_i} axes**, then **Pre-Multiply** to compute total rotation as:

$$\underline{\underline{R}} = \underline{\underline{R}}(x_2, 90^o)\underline{\underline{R}}(x_3, 90^o) = \{final - orientation\} = {}^x_y\underline{\underline{R}} =$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$



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$$\underline{\underline{R}} = \underline{\underline{R}}(x_2, 90^o)\underline{\underline{R}}(x_3, 90^o) = \{final - orientation\} = {}^{x}_{y}\underline{\underline{R}} =$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\{\underline{x}\} = \underline{\underline{R}}\{\underline{y}\}$$

$$\{New - Orientation\}_x = \underline{\underline{R}}\{Old - Orientation\}_y =$$

$$\begin{bmatrix} A' & B' & C' & D' & E' & F' \\ 0 & 0 & 2 & 0 & 0 & 2 \\ 1 & 1 & 1 & -1 & -1 & -1 \\ 0 & 4 & 0 & 0 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} A & B & C & D & E & F \\ 1 & 1 & 1 & -1 & -1 & -1 \\ 0 & 4 & 0 & 0 & 4 & 0 \\ 0 & 0 & 2 & 0 & 0 & 2 \end{bmatrix}$$

Rotation About an Arbitrary Axis (Equivalent Angle-Axis Representation):

***Euler's Theorem(10-continued)*: Any change of orientation for a rigid body with a fixed body point can be accomplished through a *General Rotation Operator* (a simple rotation) with a proper axis and angle selection.**

Consider the following coordinates:

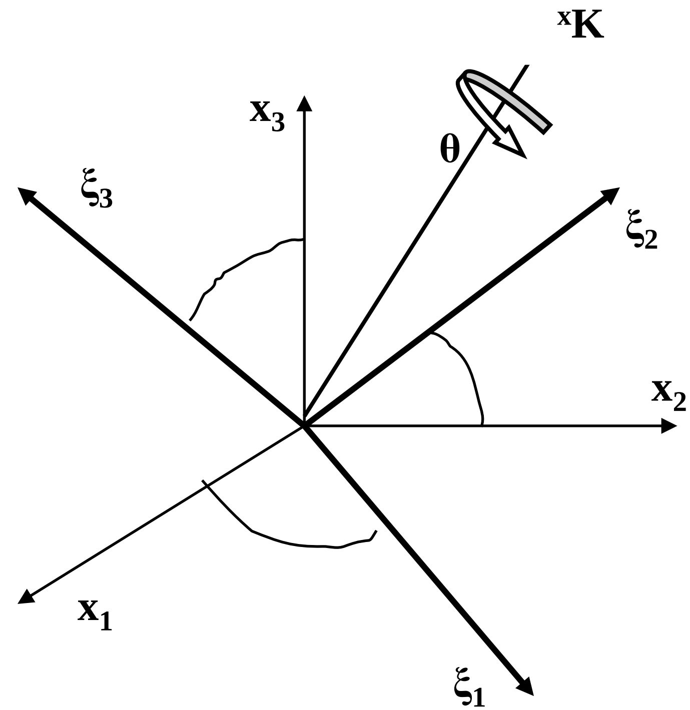
$\{x_i\}$ **: Spatial Coordinates**

$\{\xi_j\}$ **: Body Coordinates**

$$\{x_i\} = \underline{\underline{R}}\{\xi_j\}$$

$$\underline{\underline{R}} = \underline{\underline{R}}({}^x\underline{K},\theta) = {}^x_\xi\underline{\underline{R}}({}^x\underline{K},\theta)$$

= A Simple/General Rotation Operator about an arbitrary axis.



Where:

$$\underline{\underline{R}}({}^{x}\underline{K},\theta)=\begin{bmatrix} k_{x1}k_{x1}v\theta+c\theta & k_{x1}k_{x2}v\theta-k_{x3}s\theta & k_{x1}k_{x3}v\theta+k_{x2}s\theta \\ k_{x1}k_{x2}v\theta+k_{x3}s\theta & k_{x2}k_{x2}v\theta+c\theta & k_{x2}k_{x3}v\theta-k_{x1}s\theta \\ k_{x1}k_{x3}v\theta-k_{x2}s\theta & k_{x2}k_{x3}v\theta+k_{x1}s\theta & k_{x3}k_{x3}v\theta+c\theta \end{bmatrix}$$

And;

$${}^{x}\underline{K}=k_{x1}\underline{e}_1+k_{x2}\underline{e}_2+k_{x3}\underline{e}_3=\begin{bmatrix}k_{x1} & k_{x2} & k_{x3}\end{bmatrix}^t \quad and \quad k_{x1}^2+k_{x2}^2+k_{x3}^2=1$$

$$v\theta=vers\theta=(1-\cos\theta)$$

<u>Ex</u>:

$$\underline{\underline{R}}(x_1,\theta)=\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \quad is \quad \underline{\underline{R}}({}^{x}\underline{K},\theta) \quad where: k_{x1}=1, \quad k_{x2}=0, \quad k_{x3}=0$$

For a given Rotation Matrix like $\underline{\underline{R}} = {}^{x}_{\xi}\underline{\underline{R}}({}^{x}\underline{K}, \theta) = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$ **one**

Can Determine *the equivalent angle-axis* by taking an inverse approach, such that:

$$\sin\theta = \pm\frac{1}{2}\sqrt{(r_{32}-r_{23})^2+(r_{13}-r_{31})^2+(r_{21}-r_{12})^2}$$

, and

$$\cos\theta = \frac{r_{11}+r_{22}+r_{33}-1}{2}$$

, where:

$$\theta = \tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right)$$

$${}^{x}\underline{K} = \frac{1}{2\sin\theta}\begin{bmatrix} r_{32}-r_{23} \\ r_{13}-r_{31} \\ r_{21}-r_{12} \end{bmatrix} = \begin{bmatrix} k_{x1} \\ k_{x2} \\ k_{x3} \end{bmatrix}$$

This solution is valid for ($0 \prec \theta \prec 180$), and for every pair of equivalent angle-axis $({}^{x}\underline{K},\theta)$, there exists another pair as $(-{}^{x}\underline{K},-\theta)$ representing the same orientation in space with the same rotation matrix. (no solutions for θ=0 and 180).

Any combination of Rotations is always equivalent to a single rotation about some axis "K" by an angle "θ".

Ex: Let

$$\underline{\underline{R}} = \underline{\underline{R}}(x_2,90)\underline{\underline{R}}(x_3,90) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

From above equations we have:

$$\sin\theta = \pm\sqrt{(1-0)^2 + (1-0)^2 + (1-0)^2} = \pm\frac{\sqrt{3}}{2}$$

$$\cos\theta = \frac{0+0+0-1}{2} = \frac{-1}{2}$$

$$\theta = \tan^{-1}\left(\frac{\pm\sqrt{3}/2}{-1/2}\right) = \pm 120^\circ$$

$$\underline{K} = \frac{1}{\sqrt{3}}\underline{e}_1 + \frac{1}{\sqrt{3}}\underline{e}_2 + \frac{1}{\sqrt{3}}\underline{e}_3, \textit{and} \quad -\underline{K} = -\left(\frac{1}{\sqrt{3}}\underline{e}_1 + \frac{1}{\sqrt{3}}\underline{e}_2 + \frac{1}{\sqrt{3}}\underline{e}_3\right)$$

$$\underline{\underline{R}} = \underline{\underline{R}}(x_2, 90)\underline{\underline{R}}(x_3, 90) = \underline{\underline{R}}(\underline{K}, 120) = \underline{\underline{R}}(-\underline{K}, -120)$$

متشکرم

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