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RIGID BODY KINEMATICS

Purpose:

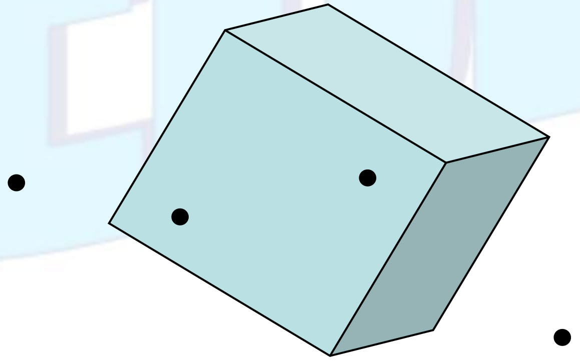
- **Analytical Description of Rigid Body Motion.**
- **Matrix Transforms to Represent Rigid Body Motion.**
- **Reinforcement of Elementary Kinematical Equations.**

Topics:

- **Translation of Rigid Bodies.**
- **Rotation of Rigid Bodies.**
- **General Motion of Rigid Bodies (i.e. Robot Kinematics)**
- **Coordinate Transformations**

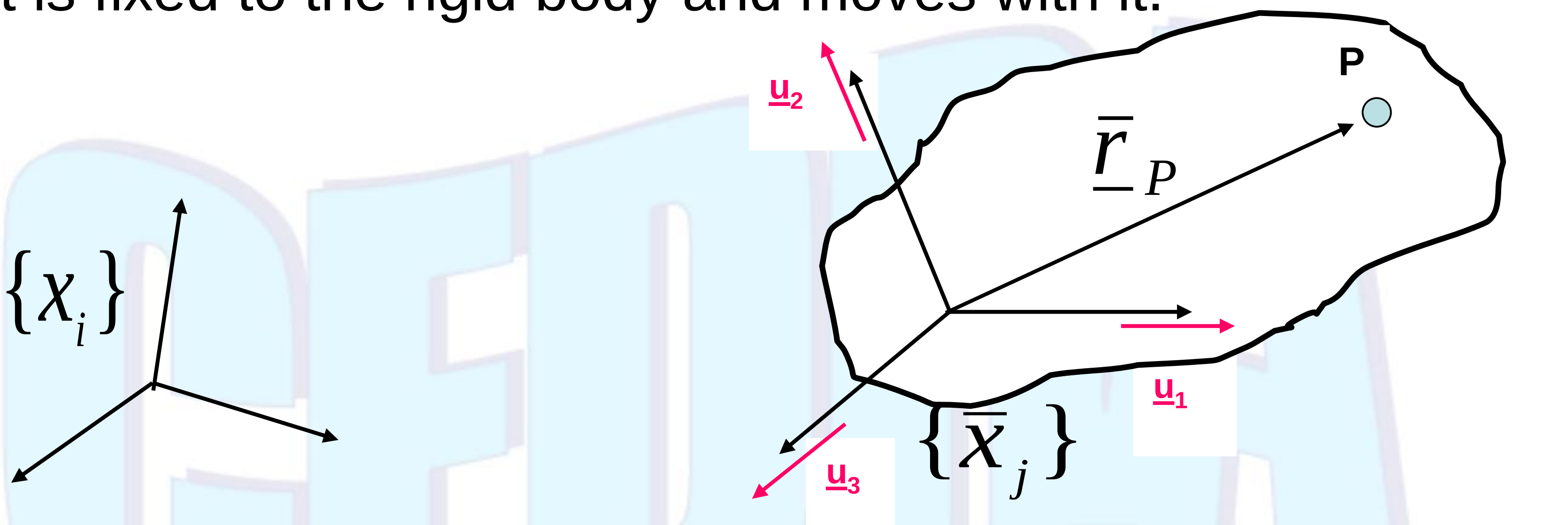
***Rigid Body*: The simplest form of a continuum, is an aggregate of particles of which the distance between any pair remains constant throughout the dynamic process.**

***Body Point*: A point which is fixed in a rigid body or its imaginary extension throughout the motion.**



Spatial Coordinate System: A coordinate system $\{X_i\}$, that is fixed in space in which the rigid body moves.

Rigid Body Coordinate System: A coordinate system $\{\overline{x}_j\}$, that is fixed to the rigid body and moves with it.

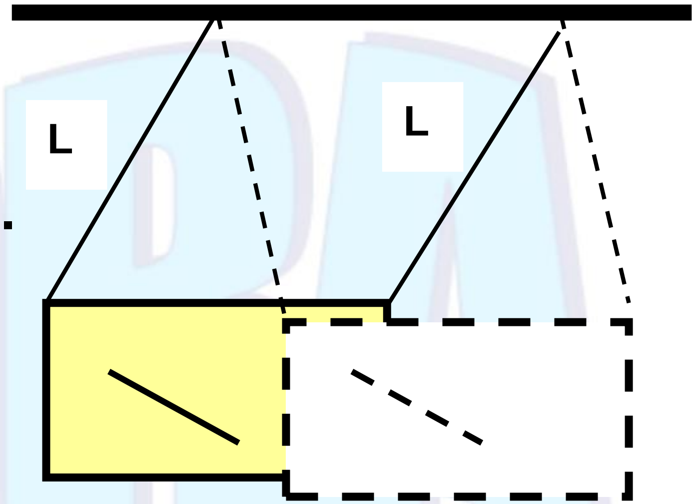


Note: **The position of a body point in its body coordinate is invariant.**

$\underline{\overline{r}}_P = \overline{x}_j \underline{u}_j$ **= (*An Invariant Set*), therefore:** **(4.1)**

$\{\dot{\overline{x}}_j\} = \{0\}$ **= (*Velocity of body points relative to the body coordinate is zero*)**

Translation: **When the line segment connecting any pair of the body points in a moving rigid body maintains its orientation during the motion, we say that the rigid body is in *pure translation*.**



Ex: ***Rigid Body in Translation***,
But particles may be in circular motion.

Theorem-7: **In a translating rigid body, the displacement of all body points are the same.**

Theorem-8: **All body points in a translating rigid body have simultaneous equal velocities and equal accelerations.**

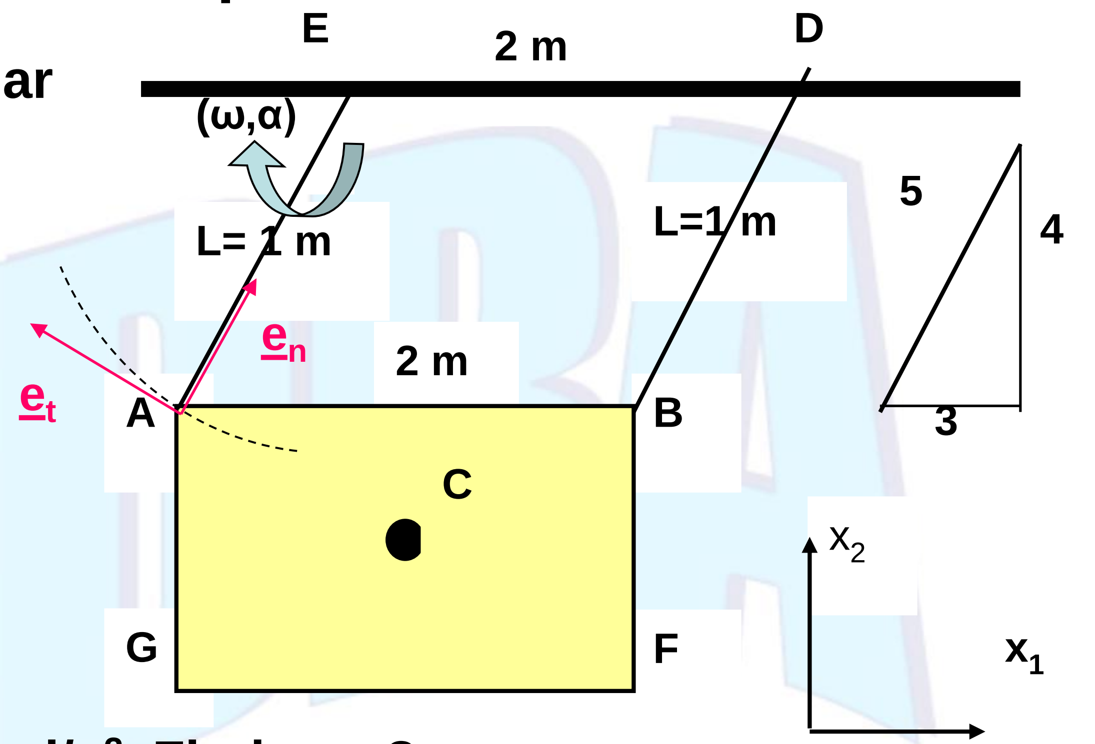
Result: In kinematical analysis, once you are sure that a rigid body has invariant orientation, any convenient body point may be selected to compute its motion.

Ex: Consider the rectangular block shown:

ABED : Parallelogram

$$\underline{\omega}_{Block} = \underline{\alpha}_{Block} = 0$$

(block stays always horizontal, therefore it is in pure translation)



Given: ω = 20 rad/s, α = 300 rad/s^2, Find $\underline{a}_c$ = ?

Solution: $$\underline{a}_C = \underline{a}_A = R\alpha\underline{e}_t + R\omega^2\underline{e}_n = 300(\frac{-4}{5}\underline{e}_1 + \frac{3}{5}\underline{e}_2) + 400(\frac{3}{5}\underline{e}_1 + \frac{4}{5}\underline{e}_2) = 500\underline{e}_2 m/s^2$$

By: Professor Ali Meghdari

Rotation: When a moving rigid body has a fixed (or momentarily fixed) body point in space, the rigid body is said to be in ***rotation***.

Theorem-9: In a rotating rigid body, when more than one fixed body points are present, these points must lie on a common straight line called the ***fixed (instantaneous) axis of rotation*** of rigid body.

We previously showed that: ***Rotation is an Orthogonal Transform***. If $\{x_i\}$ is a spatial coordinate system, and $\{\bar{x}_j\}$ is the rigid body coordinate system, we have:

$$\{x_i^P\} = \underline{\underline{R}}\{\bar{x}_j^P\} \qquad (4.2)$$

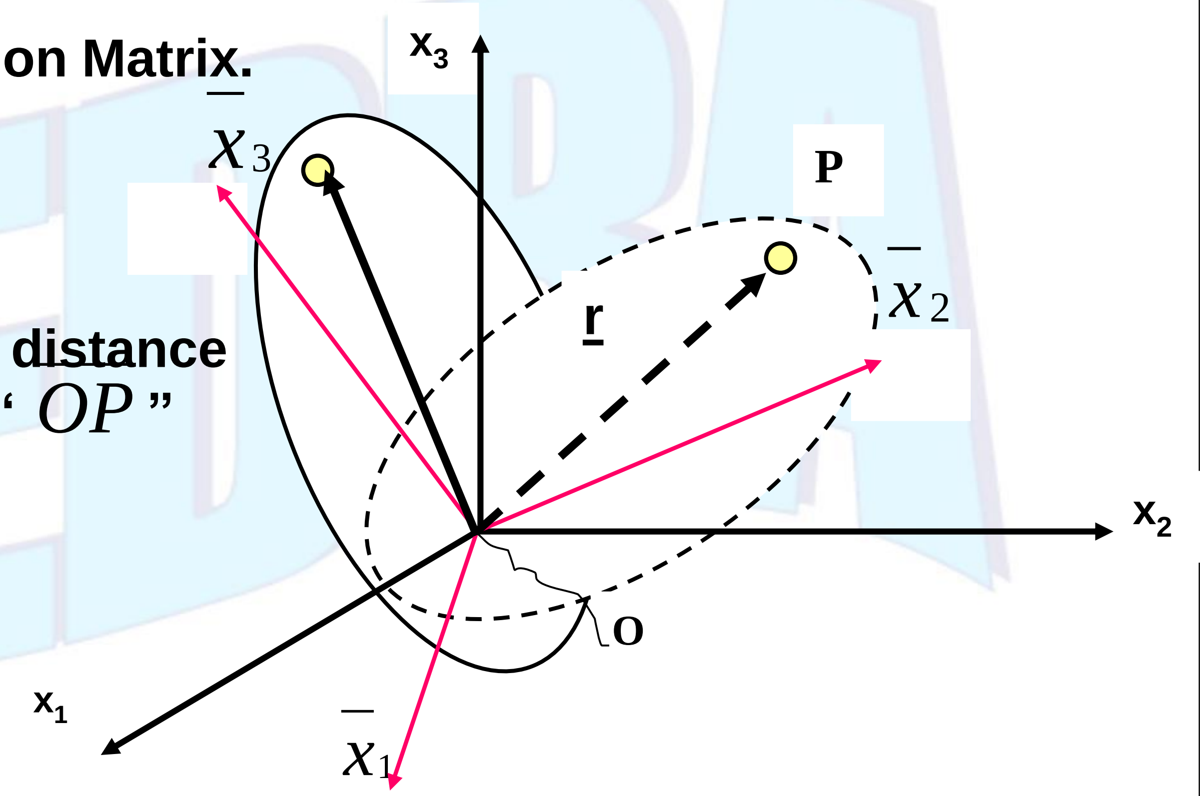
$$\{x_i^P\} = \underline{\underline{R}}\{\bar{x}_j^P\} \tag{4.2}$$

where: $\{x_i^P\}$: **Spatial position of "P" after rotation.**

$\{\bar{x}_j^P\}$: **Original position of "P".**

$\underline{\underline{R}}$: **Rotation Matrix.**

Note: For a rigid body, the distance between two body points " OP **" is *constant*.**



$$\overline{OP} = |\underline{r}| = \underline{r}\cdot\underline{r} = \{x_i^P\}^t\{x_i^P\} = [x_1 \quad x_2 \quad x_3]\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

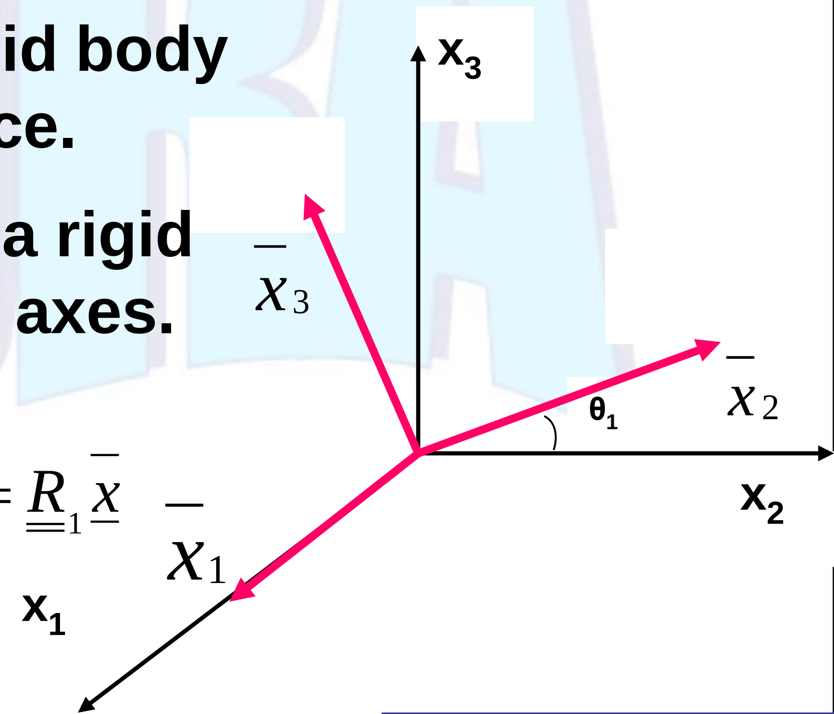
$$\{x_i^P\}^t = \{\bar{x}_j^P\}^t\,\underline{\underline{R}}^t$$

$$\underline{r}\cdot\underline{r} = \{\bar{x}_j^P\}^t\,\underline{\underline{R}}^t\,\underline{\underline{R}}\{\bar{x}_j^P\} = \{\bar{x}_j^P\}^t\{\bar{x}_j^P\} = \{x_i^P\}^t\{x_i^P\}$$ = ***constant***

Simple Rotation: **Rotation of a rigid body about a general fixed axes in space.**

Elementary Rotation: **Rotation of a rigid body about one of the coordinate axes.**

$$\underline{\underline{R}}_1(x_1,\theta_1) = \begin{bmatrix}1 & 0 & 0\\ 0 & \cos\theta_1 & -\sin\theta_1\\ 0 & \sin\theta_1 & \cos\theta_1\end{bmatrix} \Rightarrow \underline{x} = \underline{\underline{R}}_1\,\underline{\bar{x}}$$



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Euler's Theorem(10): **Any change of orientation (about an arbitrary axis) for a rigid body with a fixed body point can be accomplished through a *simple rotation* [Eq. (4.2)]. Then, the rigid body rotation can be resolved into three *elementary rotations*, where the angles of these rotations are called the *Euler's Angles*.**

Finite (Spatial) Rotation: **A spatial rotation features rotation about two or more nonparallel coordinate axes. Note that finite rotation is order dependent, and *does not satisfy the Commutative Law*.**

***Two situations commonly arise in sequential rotations*:**

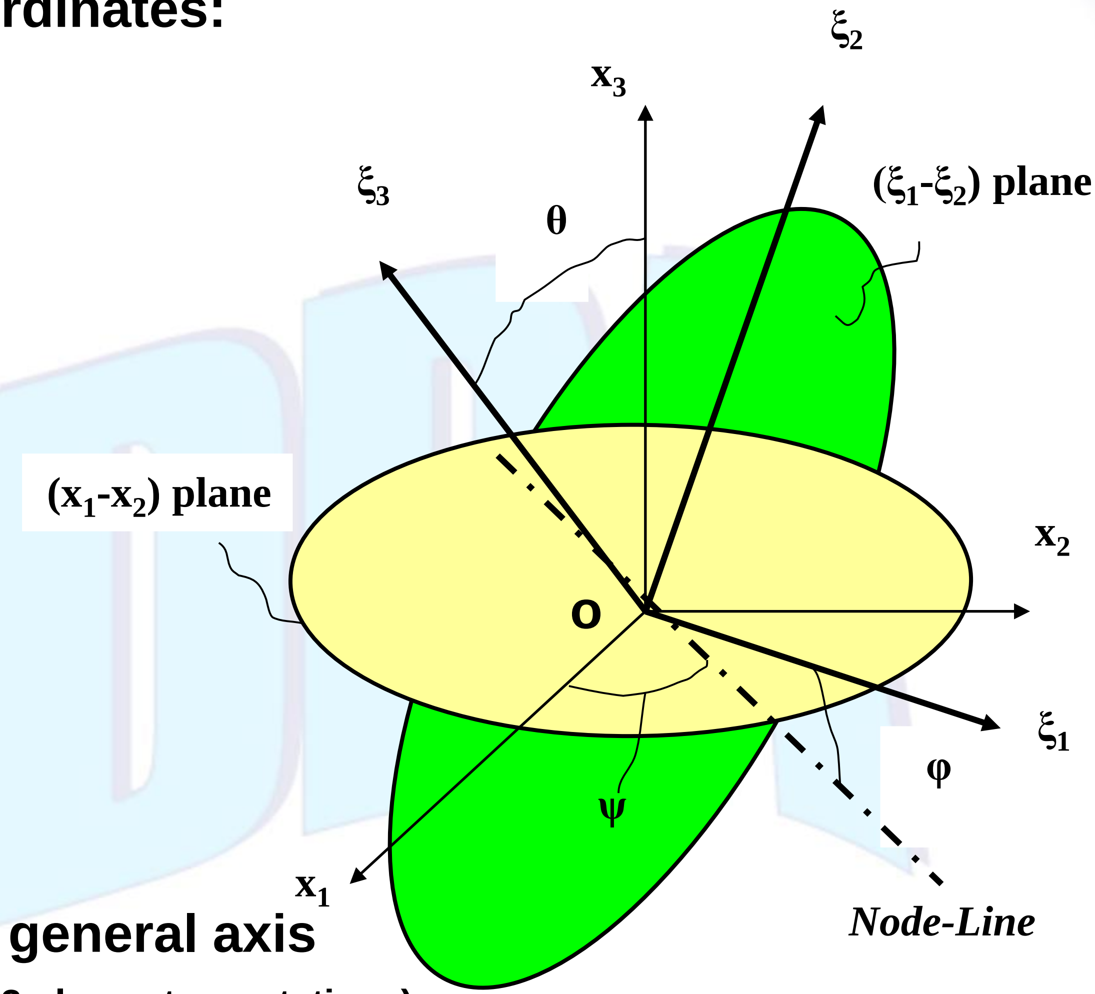
1. Body Fixed Rotations (*Rotations about New-axis*)
2. Space Fixed Rotations (*Rotations about Original/Old-axis*)

1. Body Fixed Rotations (*Rotations about New-axis*)

Consider the following coordinates:

$\{x_i\}$: Spatial Coordinates

$\{\xi_j\}$: Body Coordinates

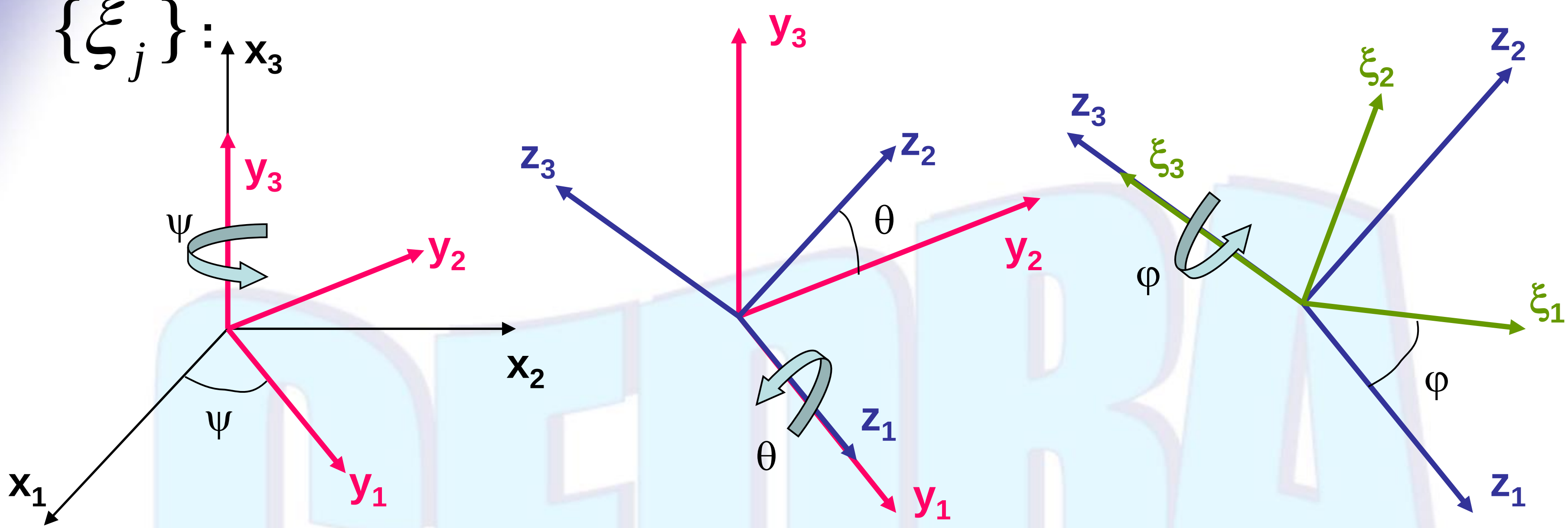


$$\{x_i\} = \underline{\underline{R}}\{\xi_j\}$$

$\underline{\underline{R}}$ **: Simple Rotation about a general axis in space (can be resolved into 3-elementary rotations).**

Let us start from $\{X_i\}$ and with a sequence of three rotations obtain

$\{\xi_j\}$:



$$\underline{x} = \underline{\underline{R}}_{\psi}\, \underline{y} \qquad \underline{y} = \underline{\underline{R}}_{\theta}\, \underline{z} \qquad \underline{z} = \underline{\underline{R}}_{\varphi}\, \underline{\xi}$$

where:

$$\underline{\underline{R}}(x_3,\psi) = \begin{bmatrix} c\psi & -s\psi & 0 \\ s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \underline{\underline{R}}(y_1,\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta & -s\theta \\ 0 & s\theta & c\theta \end{bmatrix}, \quad \underline{\underline{R}}(z_3,\varphi) = \begin{bmatrix} c\varphi & -s\varphi & 0 \\ s\varphi & c\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore:

$$\underline{x} = \underline{\underline{R}}_{\psi}\,\underline{y} = \underline{\underline{R}}_{\psi}\,\underline{\underline{R}}_{\theta}\,\underline{z} = \underline{\underline{R}}_{\psi}\,\underline{\underline{R}}_{\theta}\,\underline{\underline{R}}_{\varphi}\,\underline{\xi} \quad \Rightarrow \quad \underline{x} = \underline{\underline{R}}\,\underline{\xi} \qquad \textbf{(4.3)}$$

where:

$$\underline{\underline{R}} = {}^{x}_{\xi}\underline{\underline{R}} = \underline{\underline{R}}_{\psi}\,\underline{\underline{R}}_{\theta}\,\underline{\underline{R}}_{\varphi} = \begin{bmatrix} c\psi c\varphi - s\psi c\theta s\varphi & -c\psi s\varphi - s\psi c\theta c\varphi & s\psi s\theta \\ s\psi c\varphi + c\psi c\theta s\varphi & -s\psi s\varphi + c\psi c\theta c\varphi & -c\psi s\theta \\ s\theta s\varphi & s\theta c\varphi & c\theta \end{bmatrix} \qquad \textbf{(4.4)}$$

or:

$$\underline{x} = {}^{x}_{\xi}\underline{\underline{R}}\,\underline{\xi} = (\underline{\underline{R}}_{1}\,\underline{\underline{R}}_{2}\,\underline{\underline{R}}_{3})\,\underline{\xi} \qquad \textbf{(4.5)}$$

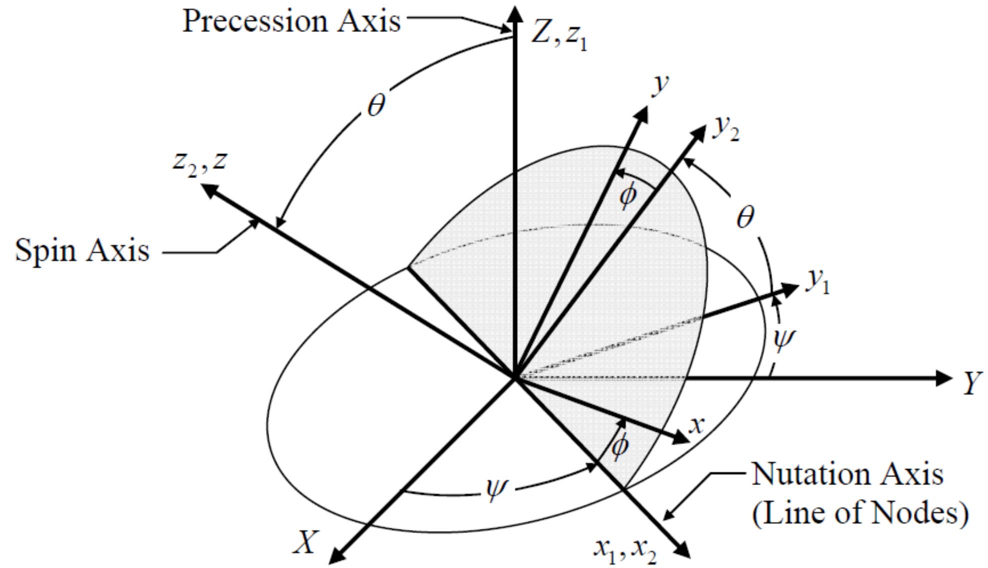
$$\underline{\underline{R}}_{1} = \underline{\underline{R}}_{\psi}, \qquad \underline{\underline{R}}_{2} = \underline{\underline{R}}_{\theta}, \qquad \underline{\underline{R}}_{3} = \underline{\underline{R}}_{\varphi}$$

Therefore, for "n" rotations:

$$\underline{\underline{R}} = \underline{\underline{R}}_{1}\,\underline{\underline{R}}_{2}\,\underline{\underline{R}}_{3}\ldots\underline{\underline{R}}_{n} \qquad \textbf{(4.6)}$$

= (*Post-Multiplication*)

It is customary to refer to these angles as (*<u>i.e. in Spinning Top or a Gyroscope</u>*):

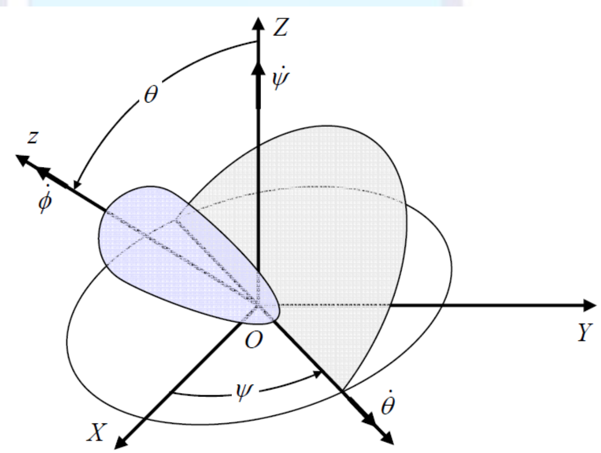


ψ : *Angle of Precession*

θ : *Angle of Nutation*

φ : *Spin Angle*

As a follow up to Euler's Theorem, angles "ψ, θ, φ" are called the *Euler's Angles*.

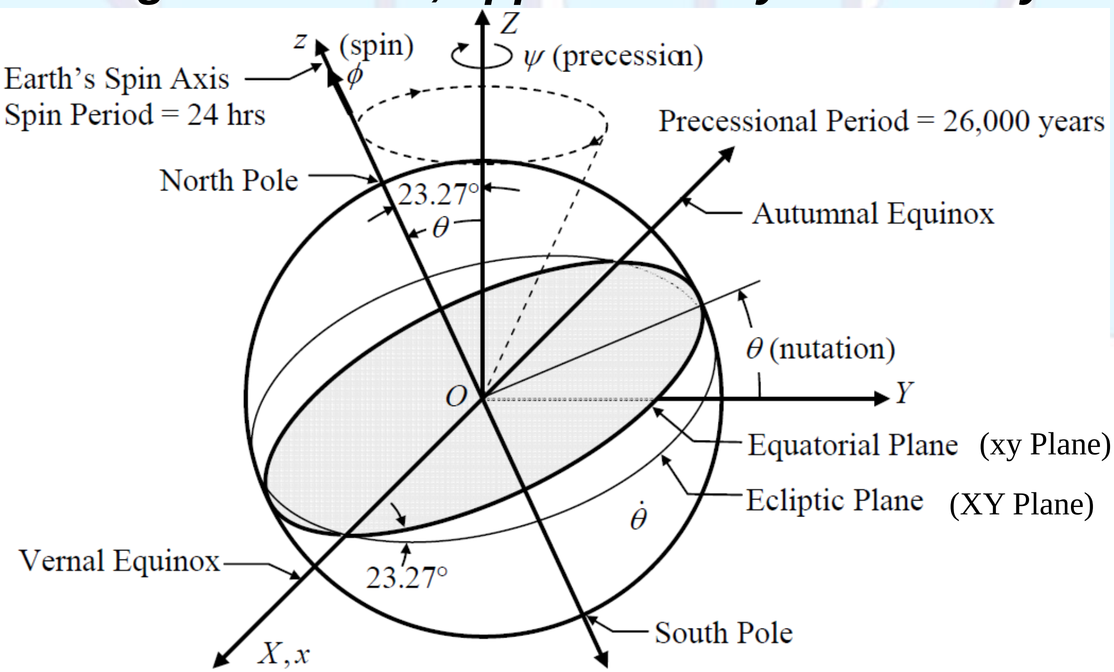


Ex: Describing the Earth's Motions in Space. In addition to its movement about the Sun in an Ecliptic orbit in the Ecliptic plane, the planet earth experiences Precession, Nutation, and Spin.

***Spin*: Daily rotation in 24 hours**

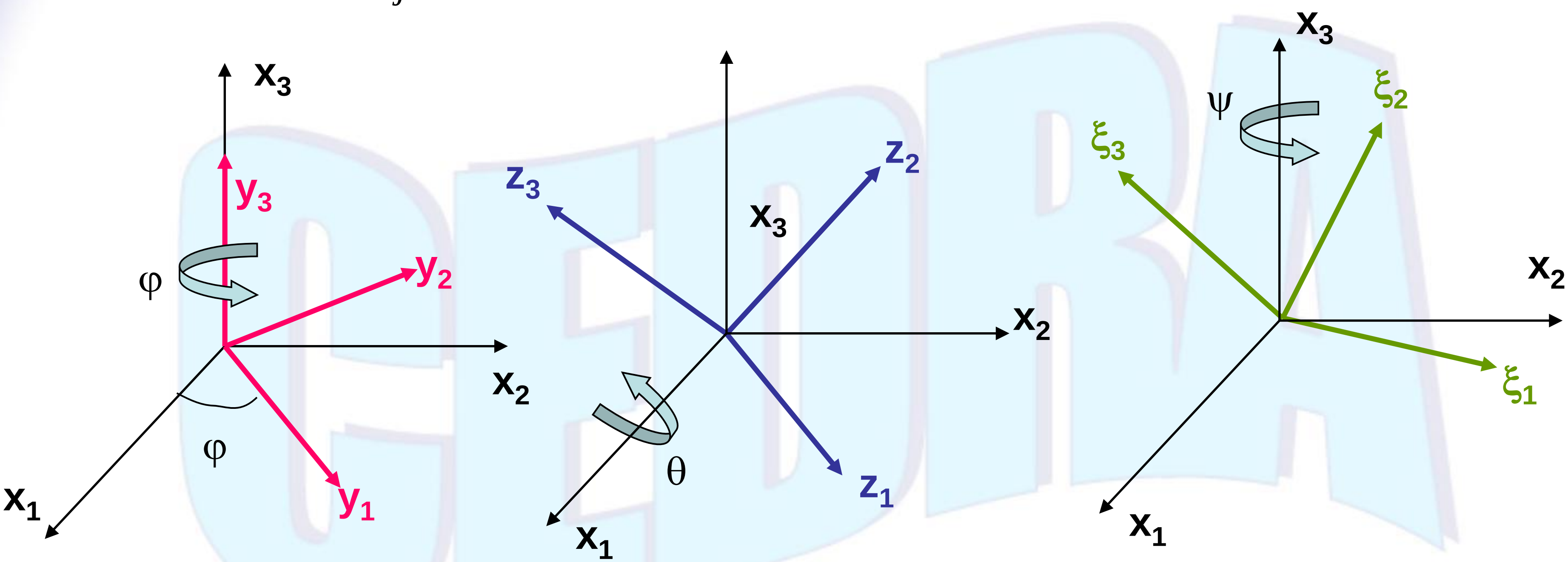
***Nutation*: Inclination of the earth's spin axis with respect to the normal to the ecliptic plane (23.27°).**

***Precession*: Earth's Spin-axis precesses along the surface of a hypothetical cone with apex angle of 23.27°, approximately once every 26000 years.**



2. Space Fixed Rotations (*Rotations about Original/Old-axis*)

Let us repeat a similar example, but this time start from $\{X_i\}$ and with a sequence of three rotations about the *original/old axis* obtain $\{\xi_j\}$:



It can be shown that (*proof in the Ginsberg's book*):

$$\underline{x} = {}^{x}_{\xi}\underline{\underline{R}}\,\underline{\xi} = \underline{\underline{R}}_{\psi}\,\underline{\underline{R}}_{\theta}\,\underline{\underline{R}}_{\varphi}\,\underline{\xi} = \underline{\underline{R}}_{3}\,\underline{\underline{R}}_{2}\,\underline{\underline{R}}_{1}\,\underline{\xi}$$ **; where: (4.7)**

$$\underline{\underline{R}}_{1} = \underline{\underline{R}}_{\varphi} = \underline{\underline{R}}(x_3,\varphi); \quad \underline{\underline{R}}_{2} = \underline{\underline{R}}_{\theta} = \underline{\underline{R}}(x_1,\theta); \quad \underline{\underline{R}}_{3} = \underline{\underline{R}}_{\psi} = \underline{\underline{R}}(x_3,\psi)$$

Therefore, for “n” rotations:

$$\underline{\underline{R}} = \underline{\underline{R}}_{n} \ldots \underline{\underline{R}}_{3}\,\underline{\underline{R}}_{2}\,\underline{\underline{R}}_{1}$$ **= (*Pre-multiplication*) (4.8)**

(Note that my conventions for Pre & Post Multiplications are opposite to that of the Ginsberg’s book, since I am defining ${}^{x}_{\xi}\underline{\underline{R}}$ *, but he is defining* ${}^{\xi}_{x}\underline{\underline{R}} \equiv {}^{x}_{\xi}\underline{\underline{R}}^{t} \equiv {}^{x}_{\xi}\underline{\underline{R}}^{-1}$*).*

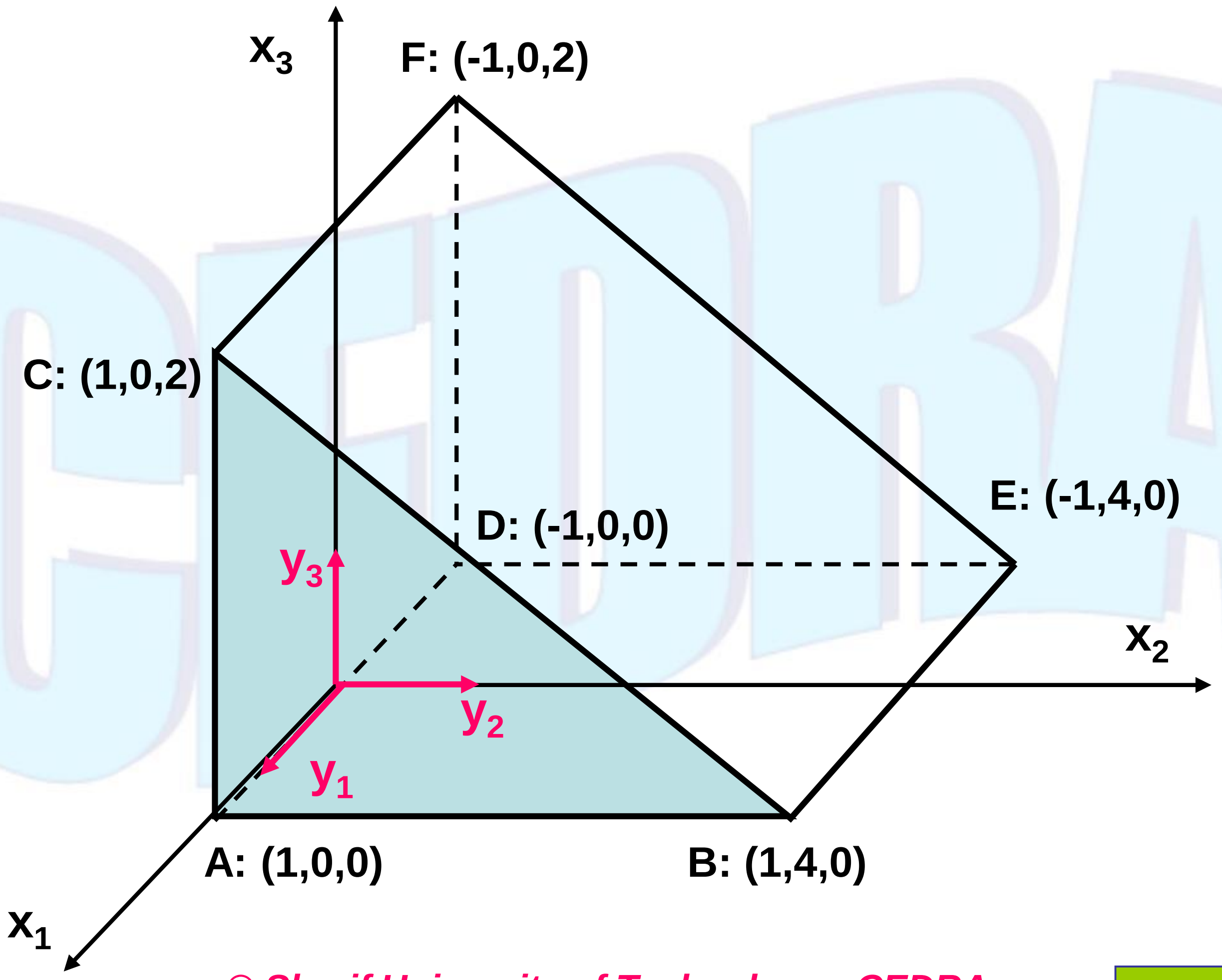
Special Case: A sequence of rotations described by $\underline{\underline{R}}_1$ and $\underline{\underline{R}}_2$ about Body Fixed (New) axes, followed by $\underline{\underline{R}}_3$ about a Space Fixed (Original) axis, and then $\underline{\underline{R}}_4$ about a Body Fixed (New) axis, would lead to:

$$\underline{\underline{R}} = \underline{\underline{R}}_3 \underline{\underline{R}}_1 \underline{\underline{R}}_2 \underline{\underline{R}}_4 \qquad and \qquad \underline{\underline{R}}^t = \underline{\underline{R}}_4^t \underline{\underline{R}}_2^t \underline{\underline{R}}_1^t \underline{\underline{R}}_3^t \qquad (4.9)$$

Note: The final orientation of a coordinate system depends on the sequence in which rotations occur, as well as the magnitude of the individual rotations and the orientation of their respective axes.

Important: Finite spatial rotations cannot be represented as vectors, because vector addition is independent of the order of addition.

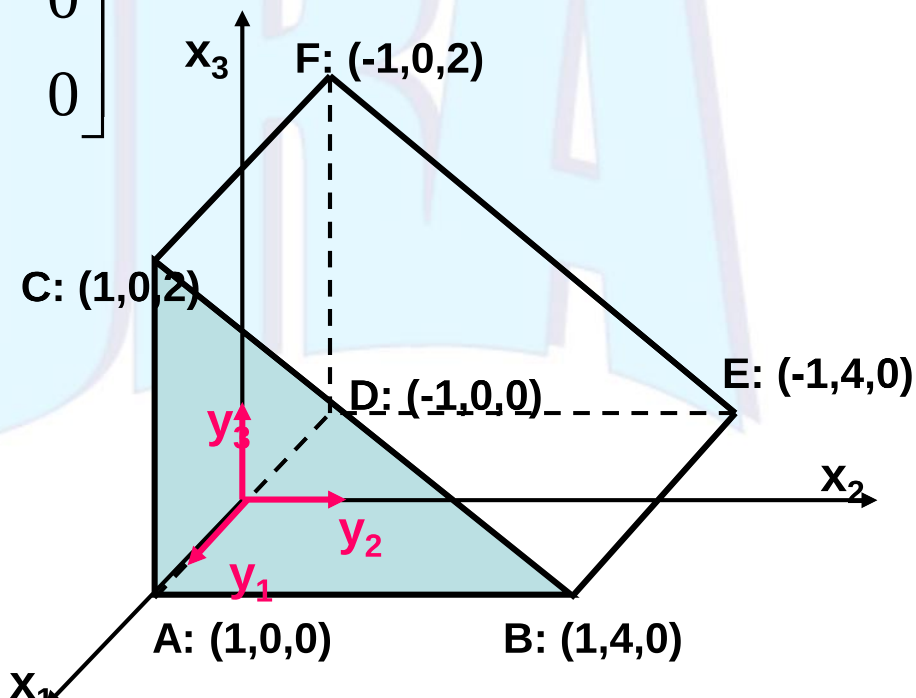
Ex: **Consider the object shown. First rotate it about the x_3-axis by 90°, and then rotate it about the x_2-axis by 90°. Determine the new orientation of the object?**



Solution: **{y} and {x} originally coincide. Since all rotations are about the Original (fixed) {x_i} axes, then Pre-Multiply to compute total rotation as:**

$$\underline{\underline{R}} = \underline{\underline{R}}(x_2, 90^o)\,\underline{\underline{R}}(x_3, 90^o) = \{\,final - orientation\,\} = {}^{x}_{y}\underline{\underline{R}} =$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$



$$\underline{\underline{R}} = \underline{\underline{R}}(x_2, 90^o)\underline{\underline{R}}(x_3, 90^o) = \{\,final - orientation\} = {}^{x}_{y}\underline{\underline{R}} =$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\{\underline{x}\} = \underline{\underline{R}}\{\underline{y}\}$$

$$\{New - Orientation\}_x = \underline{\underline{R}}\{Old - Orientation\}_y =$$

$$\begin{bmatrix} A' & B' & C' & D' & E' & F' \\ 0 & 0 & 2 & 0 & 0 & 2 \\ 1 & 1 & 1 & -1 & -1 & -1 \\ 0 & 4 & 0 & 0 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} A & B & C & D & E & F \\ 1 & 1 & 1 & -1 & -1 & -1 \\ 0 & 4 & 0 & 0 & 4 & 0 \\ 0 & 0 & 2 & 0 & 0 & 2 \end{bmatrix}$$

Rotation About an Arbitrary Axis (Equivalent Angle-Axis Representation):

Euler's Theorem(10-continued): Any change of orientation for a rigid body with a fixed body point can be accomplished through a *General Rotation Operator* (a simple rotation) with a proper axis and angle selection.

Consider the following coordinates:

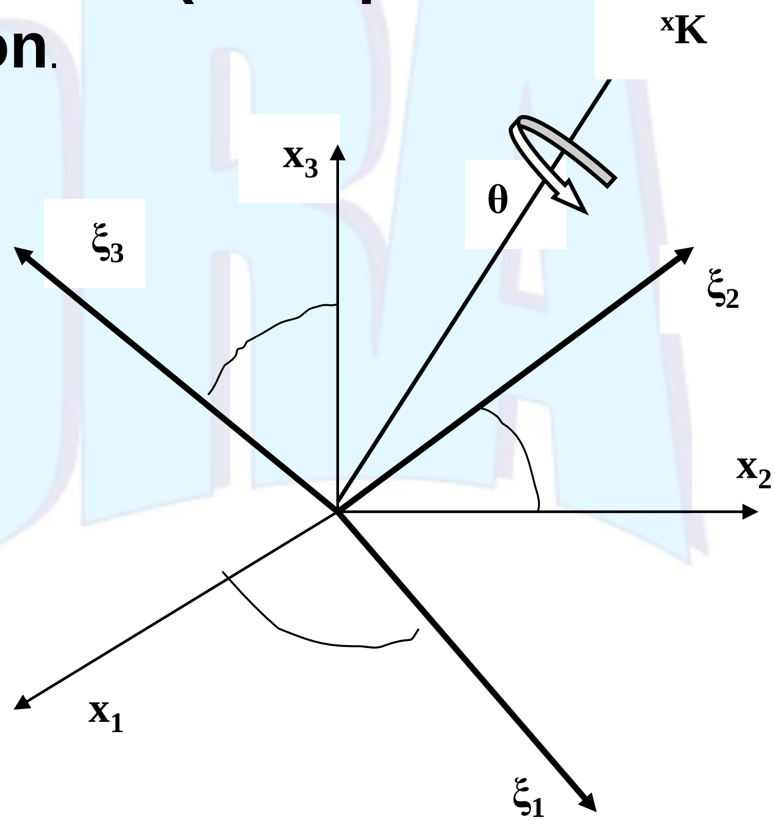
$\{x_i\}$: **Spatial Coordinates**

$\{\xi_j\}$: **Body Coordinates**

$$\{x_i\} = \underline{\underline{R}}\{\xi_j\}$$

$$\underline{\underline{R}} = \underline{\underline{R}}({}^x\underline{K},\theta) = {}^x_\xi\underline{\underline{R}}({}^x\underline{K},\theta)$$

= A Simple/General Rotation Operator about an arbitrary axis.



Where:

$$\underline{\underline{R}}({}^{x}\underline{K},\theta)=\begin{bmatrix} k_{x1}k_{x1}v\theta+c\theta & k_{x1}k_{x2}v\theta-k_{x3}s\theta & k_{x1}k_{x3}v\theta+k_{x2}s\theta \\ k_{x1}k_{x2}v\theta+k_{x3}s\theta & k_{x2}k_{x2}v\theta+c\theta & k_{x2}k_{x3}v\theta-k_{x1}s\theta \\ k_{x1}k_{x3}v\theta-k_{x2}s\theta & k_{x2}k_{x3}v\theta+k_{x1}s\theta & k_{x3}k_{x3}v\theta+c\theta \end{bmatrix}$$

And;

$$ {}^{x}\underline{K}=k_{x1}\underline{e}_1+k_{x2}\underline{e}_2+k_{x3}\underline{e}_3=\begin{bmatrix}k_{x1} & k_{x2} & k_{x3}\end{bmatrix}^t \quad and \quad k_{x1}^2+k_{x2}^2+k_{x3}^2=1$$

$$v\theta=vers\theta=(1-\cos\theta)$$

<u>Ex</u>:

$$\underline{\underline{R}}(x_1,\theta)=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta\end{bmatrix} \quad is \quad \underline{\underline{R}}({}^{x}\underline{K},\theta) \quad where: k_{x1}=1,\quad k_{x2}=0,\quad k_{x3}=0$$

For a given Rotation Matrix like $\underline{\underline{R}} = {}^{x}_{\xi}\underline{\underline{R}}({}^{x}\underline{K}, \theta) = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$**, one can**

Determine *the equivalent angle-axis* by taking an inverse approach, such that:

$$\sin\theta = \pm\frac{1}{2}\sqrt{(r_{32}-r_{23})^2+(r_{13}-r_{31})^2+(r_{21}-r_{12})^2}$$ **, and**

$$\cos\theta = \frac{r_{11}+r_{22}+r_{33}-1}{2}$$ **, where:**

$$\theta = \tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right)$$

$${}^{x}\underline{K} = \frac{1}{2\sin\theta}\begin{bmatrix} r_{32}-r_{23} \\ r_{13}-r_{31} \\ r_{21}-r_{12} \end{bmatrix} = \begin{bmatrix} k_{x1} \\ k_{x2} \\ k_{x3} \end{bmatrix}$$

This solution is valid for ($0 \prec \theta \prec 180$), and for every pair of equivalent angle-axis $({}^{x}\underline{K}, \theta)$, there exists another pair as $(-{}^{x}\underline{K}, -\theta)$ representing the same orientation in space with the same rotation matrix. (no solutions for θ=0 and 180).

<u>Any combination of Rotations is always equivalent to a single rotation about some axis</u> "K" <u>by an angle</u> "θ".

<u>Ex</u>: Let

$$\underline{\underline{R}} = \underline{\underline{R}}(x_2, 90)\underline{\underline{R}}(x_3, 90) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

From above equations we have:

$$\sin\theta = \pm\sqrt{(1-0)^2 + (1-0)^2 + (1-0)^2} = \pm\frac{\sqrt{3}}{2}$$

$$\cos\theta = \frac{0+0+0-1}{2} = \frac{-1}{2}$$

$$\theta = \tan^{-1}\left(\frac{\pm\sqrt{3}/2}{-1/2}\right) = \pm 120^\circ$$

$$\underline{K} = \frac{1}{\sqrt{3}}\underline{e}_1 + \frac{1}{\sqrt{3}}\underline{e}_2 + \frac{1}{\sqrt{3}}\underline{e}_3, and \quad -\underline{K} = -\left(\frac{1}{\sqrt{3}}\underline{e}_1 + \frac{1}{\sqrt{3}}\underline{e}_2 + \frac{1}{\sqrt{3}}\underline{e}_3\right)$$

$$\underline{\underline{R}} = \underline{\underline{R}}(x_2,90)\underline{\underline{R}}(x_3,90) = \underline{\underline{R}}(\underline{K},120) = \underline{\underline{R}}(-\underline{K},-120)$$


متشکرم