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ORTHOGONAL CURVILINEAR COORDINATES

Description specifies the position of a point, by giving the value of *3-parameters*, q^{α} , (i.e. θ, φ, R) which form an *orthogonal mesh* in space.

There exist a unique transformation between the Cartesian Coordinates (x, y, z) and the Orthogonal Coordinates, q^{α} , (i.e. θ, φ, R), such that:

$$x = x(\theta, \varphi, R), \quad y = y(\theta, \varphi, R), \quad z = z(\theta, \varphi, R) \qquad (3.16)$$

$$\theta = \theta(x, y, z), \quad \varphi = \varphi(x, y, z), \quad R = R(x, y, z) \qquad (3.17)$$

$$x = x(\theta, \varphi, R), \quad y = y(\theta, \varphi, R), \quad z = z(\theta, \varphi, R) \qquad (3.16)$$

$$\theta = \theta(x, y, z), \quad \varphi = \varphi(x, y, z), \quad R=R(x, y, z) \qquad (3.17)$$

When *two* of the parameters of q^{α} (i.e. θ, φ, R) are held *constant* while the *third* is given a range of values, the first group of equations (16) and (17) specifies a curve in space in parametric form.

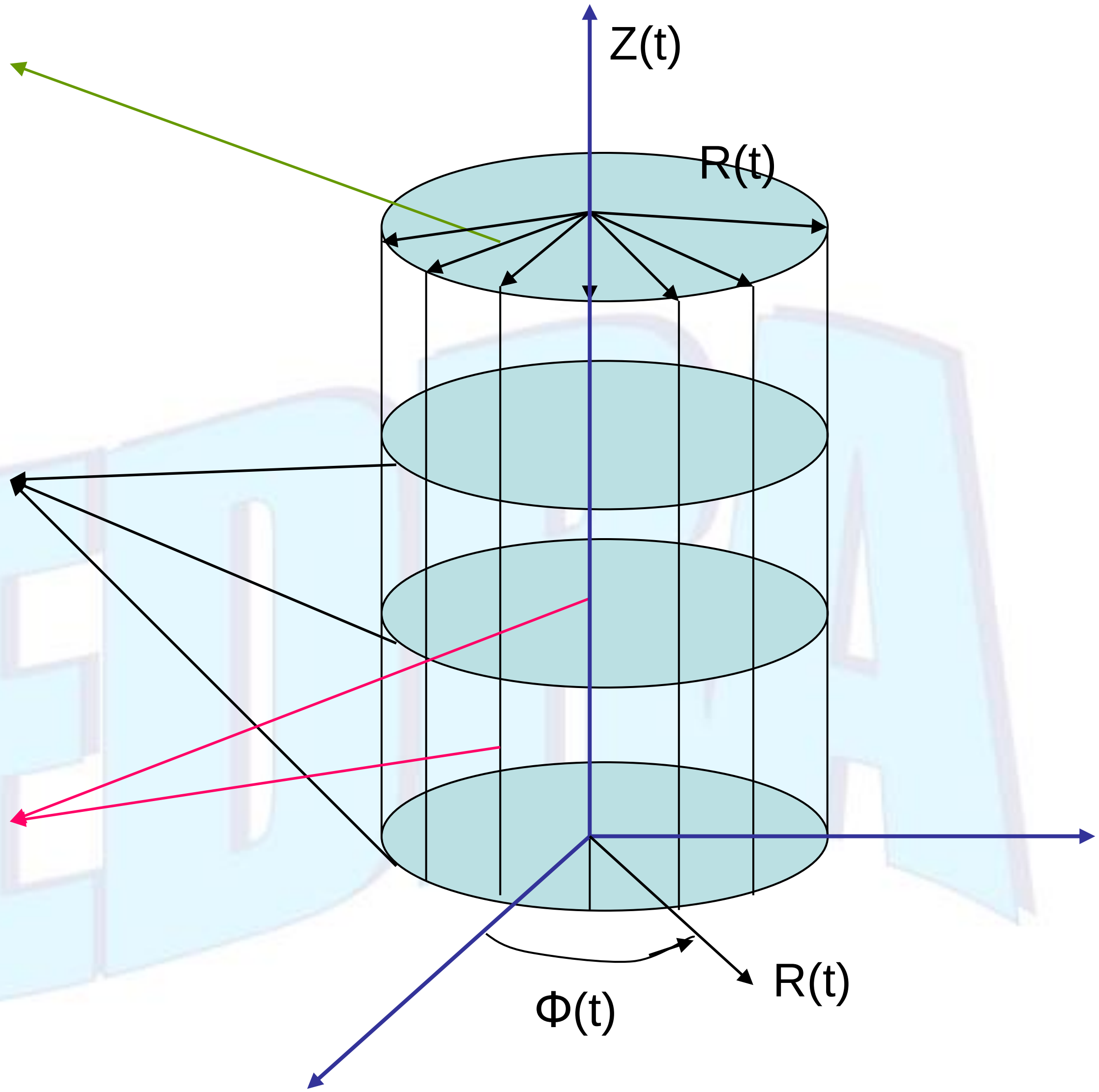
When the constant parameter pair is given a variety of values, the result is *a family of curves*. Repeating this procedure with each of the other pairs of parameters held constant, produces *two more families of curves* (i.e. *called mesh*). The families of curves are mutually *Orthogonal*.

They are named after one of the types of surfaces on which one of the curvilinear coordinates is constant.

$$\begin{cases} X = X(R, \varphi_1, Z_1) \\ X = X(R, \varphi_2, Z_2) \\ \vdots \\ X = X(R, \varphi_i, Z_i) \end{cases}$$

$$\begin{cases} X = X(R_1, \varphi, Z_1) \\ X = X(R_2, \varphi, Z_2) \\ \vdots \\ X = X(R_i, \varphi, Z_i) \end{cases}$$

$$\begin{cases} X = X(R_1, \varphi_1, Z) \\ X = X(R_2, \varphi_2, Z) \\ \vdots \\ X = X(R_i, \varphi_i, Z) \end{cases}$$



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$$\begin{cases} X = X(\theta, \varphi_1, R_1) \\ X = X(\theta, \varphi_2, R_2) \\ \vdots \\ X = X(\theta, \varphi_i, R_i) \end{cases}$$

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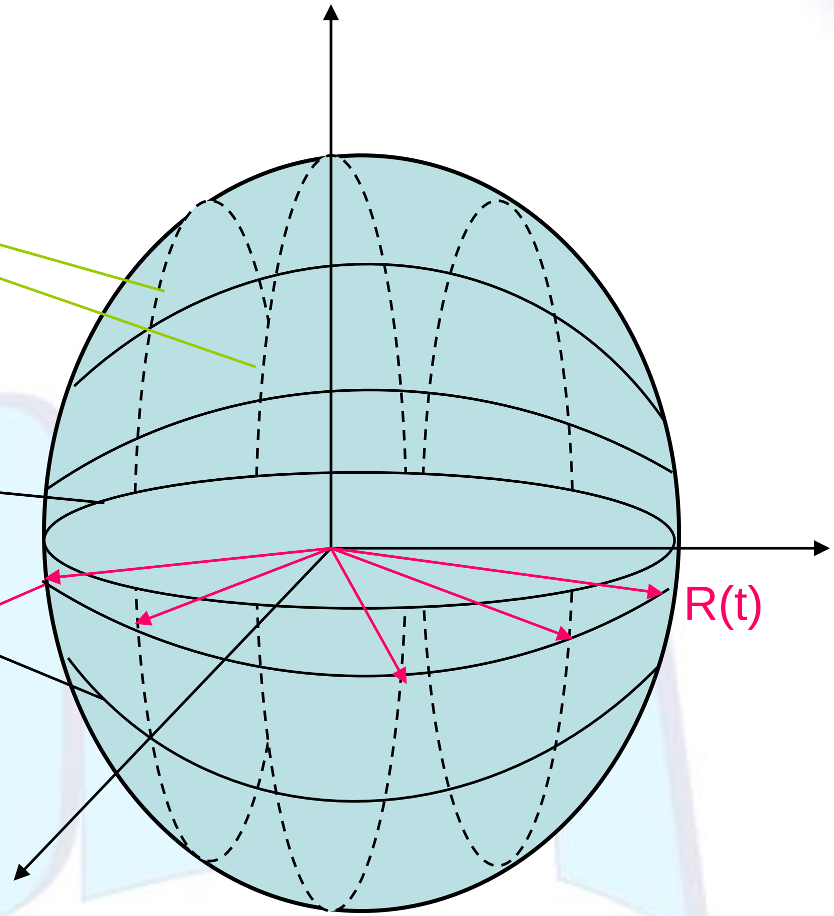


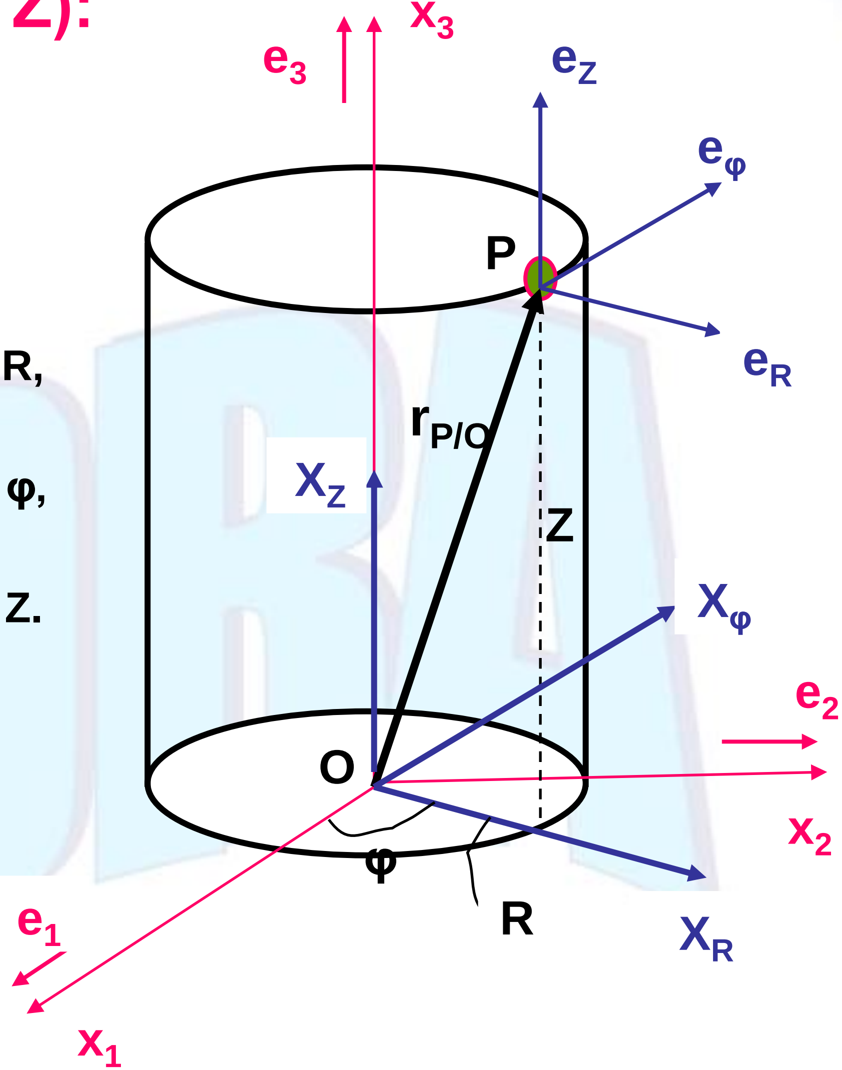
Cylindrical Coordinates (R, φ, Z):

Base Vectors **are: {$\underline{x}_R$, $\underline{x}_\varphi$, $\underline{x}_Z$}**

$\underline{e}_R$: unit vector in the direction of increasing R,

e_φ : unit vector in the direction of increasing φ,

e_Z : unit vector in the direction of increasing Z.



Position Vector:

$$\underline{r}_{P/O} = R\underline{e}_R + Z\underline{e}_Z \quad (3.18)$$

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Velocity Vector:

$$\underline{v}_P = \dot{R}\underline{e}_R + R\dot{\underline{e}}_R + \dot{Z}\underline{e}_Z + Z\dot{\underline{e}}_Z$$

unit vectors {$\underline{e}_R$, $\underline{e}_\varphi$, $\underline{e}_Z$} all rotate with an angular velocity " $\dot{\varphi}$ ",

then using *Jaumann Rate*, we have:

$$\dot{\underline{e}}_R = \underline{\omega}_R \times \underline{e}_R = (\dot{\varphi}\underline{e}_Z) \times (\underline{e}_R) = \dot{\varphi}\underline{e}_\varphi \qquad and \qquad \dot{\underline{e}}_Z = 0$$

$$\underline{v}_P = \dot{R}\underline{e}_R + R\dot{\varphi}\underline{e}_\varphi + \dot{Z}\underline{e}_Z \qquad \textbf{(3.19)}$$

Acceleration Vector:

$$\underline{a}_P = \ddot{R}\underline{e}_R + \dot{R}\dot{\underline{e}}_R + \dot{R}\dot{\varphi}\underline{e}_\varphi + R\ddot{\varphi}\underline{e}_\varphi + R\dot{\varphi}\dot{\underline{e}}_\varphi + \ddot{Z}\underline{e}_Z + \dot{Z}\dot{\underline{e}}_Z$$

$$\dot{\underline{e}}_\varphi = \underline{\omega}_\varphi \times \underline{e}_\varphi = (\dot{\varphi}\underline{e}_Z) \times (\underline{e}_\varphi) = -\dot{\varphi}\underline{e}_R \quad \text{and} \quad \dot{\underline{e}}_Z = 0$$

$$\underline{a}_P = (\ddot{R} - R\dot{\varphi}^2)\underline{e}_R + (2\dot{R}\dot{\varphi} + R\ddot{\varphi})\underline{e}_\varphi + \ddot{Z}\underline{e}_Z = a_R\underline{e}_R + a_\varphi\underline{e}_\varphi + a_Z\underline{e}_Z$$

(3.20)

- a_R : ***Radial Acceleration***
- a_φ : ***Transverse Acceleration***
- a_Z : ***Axial Acceleration***
- $2\dot{R}\dot{\varphi}$: ***Coriolis Acceleration (due*** **to the simultaneous change in "R" and "φ" with respect to time).**

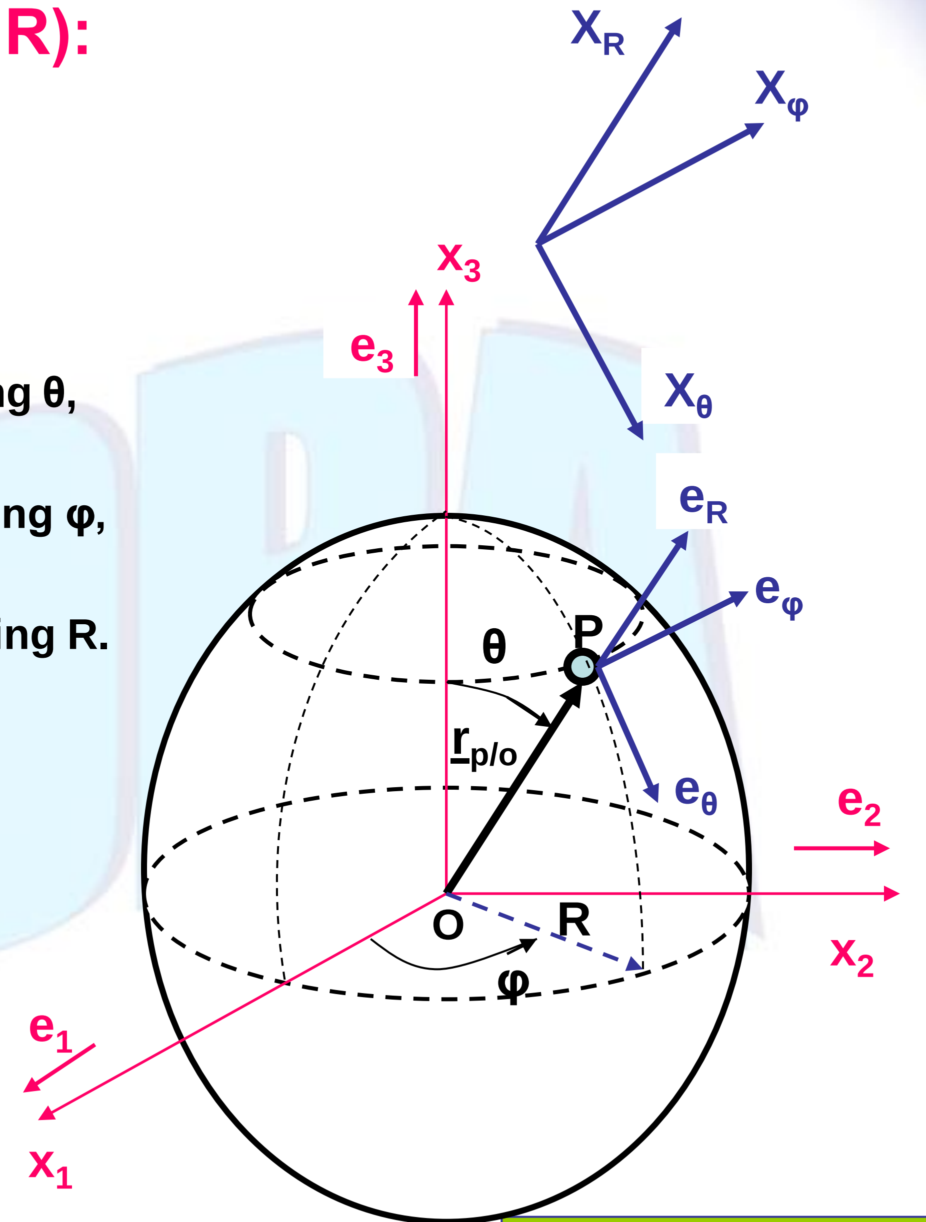
Spherical Coordinates (θ, φ, R):

Base Vectors are: $\{\underline{x}_\theta, \underline{x}_\varphi, \underline{x}_R\}$

$\underline{e}_\theta$: unit vector in the direction of increasing θ,

e_φ : unit vector in the direction of increasing φ,

e_R : unit vector in the direction of increasing R.



Position Vector:

$$\underline{r}_{P/O} = R\underline{e}_R \qquad (3.21)$$

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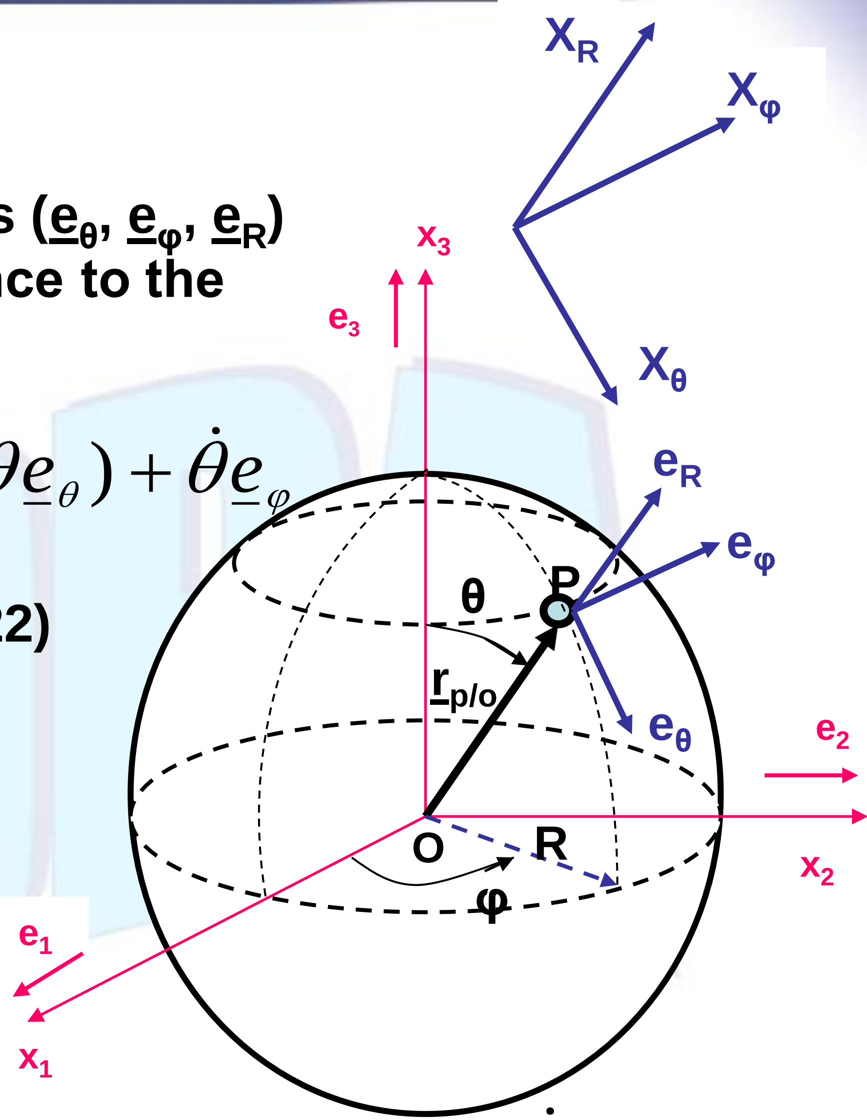
Spherical Coordinates **(θ, φ, R):**

Note that **orientation of the unit vectors ($\underline{e}_\theta$, $\underline{e}_\varphi$, $\underline{e}_R$) Changes with "θ" and "φ", in accordance to the following *angular velocity*:**

$$\underline{\Omega} = \dot{\varphi}\underline{e}_3 + \dot{\theta}\underline{e}_\varphi = \dot{\varphi}(\cos\theta\underline{e}_R - \sin\theta\underline{e}_\theta) + \dot{\theta}\underline{e}_\varphi$$

(3.22)



Now, applying *Jaumann Rate* to the position vector $\underline{r}_{P/O}$ rotating with the angular velocity $\underline{\Omega}$, we have:

***Velocity Vector*:**

$$\underline{v}_P = \dot{\underline{r}}_{P/O} = \dot{R}\underline{e}_R + \underline{\Omega}\times\underline{r}_{P/O} = \dot{R}\underline{e}_R + (R\dot{\varphi}\sin\theta\underline{e}_\varphi + R\dot{\theta}\underline{e}_\theta) \quad \textbf{(3.23)}$$

Acceleration Vector:

$$\underline{a}_P = \underline{\dot{v}}_P = (\dot{v}_P)_i \underline{e}_i + \underline{\Omega} \times \underline{v}_P = \ddot{R}\underline{e}_R + (\dot{R}\dot{\varphi}\sin\theta + R\ddot{\varphi}\sin\theta + R\dot{\theta}\dot{\varphi}\cos\theta)\underline{e}_\varphi +$$
$$(\dot{R}\dot{\theta} + R\ddot{\theta})\underline{e}_\theta + [\dot{R}\dot{\varphi}\sin\theta\underline{e}_\varphi - R\dot{\varphi}^2\sin^2\theta\underline{e}_R + \dot{R}\dot{\theta}\underline{e}_\varphi - R\dot{\theta}^2\underline{e}_R -$$
$$R\dot{\varphi}^2\sin\theta\cos\theta\underline{e}_\theta + R\dot{\theta}\dot{\varphi}\cos\theta\underline{e}_\varphi]$$

$$\underline{a}_P = (\ddot{R} - R\dot{\varphi}^2\sin^2\theta - R\dot{\theta}^2)\underline{e}_R +$$
$$(2\dot{R}\dot{\varphi}\sin\theta + 2R\dot{\theta}\dot{\varphi}\cos\theta + R\ddot{\varphi}\sin\theta)\underline{e}_\varphi +$$
$$+ (R\ddot{\theta} + 2\dot{R}\dot{\theta} - R\dot{\varphi}^2\sin\theta\cos\theta)\underline{e}_\theta$$

(3.24)

Kinematical Quantities in Curvilinear Coordinates via Transform Approach:

Theorem-6: **The orientation of a curvilinear coordinate, q^{α}, (i.e. R, φ, Z), at a point "P" in space is defined by the direction of the *Base Vectors*, $\underline{g}_{\alpha}$, at that point. If we let:**

$\underline{r}(q^{\alpha})$ **: position vector of "P", then;**

Base Vectors **=** $\underline{g}_{\alpha} = \dfrac{\partial \underline{r}}{\partial q^{\alpha}}$**, and unit vectors of $\underline{g}_{\alpha}$ are:** $\underline{e}_{\alpha} = \dfrac{\underline{g}_{\alpha}}{\left|\underline{g}_{\alpha}\right|}$

(3.25)

Note that **the curvilinear coordinates are *Orthogonal* if the base vectors form an orthogonal set, that is: "** $\underline{g}_{\alpha} \cdot \underline{g}_{\beta} = 0$ **for** $\alpha \neq \beta$ **".**

Ex: **In Cylindrical Coordinates**: $q^{\alpha} \equiv (R, \varphi, Z)$

$$\underline{r} = x_i \underline{e}_i = R\cos\varphi \underline{e}_1 + R\sin\varphi \underline{e}_2 + Z\underline{e}_3$$ **(position vector)**

Base vectors **representing the orientation of the cylindrical coordinates at point P are:**

$$\begin{cases} \underline{X}_R = \underline{g}_R = \dfrac{\partial \underline{r}}{\partial R} = \cos\varphi \underline{e}_1 + \sin\varphi \underline{e}_2 \\ \underline{X}_\varphi = \underline{g}_\varphi = \dfrac{\partial \underline{r}}{\partial \varphi} = -R\sin\varphi \underline{e}_1 + R\cos\varphi \underline{e}_2 \\ \underline{X}_Z = \underline{g}_Z = \dfrac{\partial \underline{r}}{\partial Z} = \underline{e}_3 \end{cases}$$

{A coordinate system is said to be *curvilinear* if one or more of its base vectors are functions of position.}

and;

$$
\begin{cases}
\underline{e}_R = \dfrac{\underline{g}_R}{\left|\underline{g}_R\right|} = \dfrac{\underline{g}_R}{1} = \underline{g}_R = \cos\varphi\,\underline{e}_1 + \sin\varphi\,\underline{e}_2 \\
\underline{e}_\varphi = \dfrac{\underline{g}_\varphi}{\left|\underline{g}_\varphi\right|} = \dfrac{\underline{g}_\varphi}{R} = -\sin\varphi\,\underline{e}_1 + \cos\varphi\,\underline{e}_2 \\
\underline{e}_Z = \dfrac{\underline{g}_Z}{\left|\underline{g}_Z\right|} = \dfrac{\underline{g}_Z}{1} = \underline{g}_Z = \underline{e}_3
\end{cases}
$$

Therefore, *Position Vector Components* of point P is represented by:

$$
\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \underline{\underline{T}}_\varphi \begin{Bmatrix} X_R \\ X_\varphi \\ X_Z \end{Bmatrix}; where: \quad \underline{\underline{T}}_\varphi = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ or:}
$$

$$\{x_i\} = \underline{\underline{T}}_{\varphi}\{x_\alpha\}$$ **, where:** $$\{x_\alpha\} = \begin{Bmatrix} X_R \\ X_\varphi \\ X_Z \end{Bmatrix} = \begin{Bmatrix} R \\ 0 \\ Z \end{Bmatrix} \qquad \textbf{(3.26)}$$

and;

$$\{x_\alpha\} = \underline{\underline{T}}^t_{\varphi}\{x_i\} \quad \Rightarrow \quad \{\underline{v}_\alpha\} = \underline{\underline{T}}^t_{\varphi}\{\underline{v}_i\} \quad \Rightarrow \quad \{a_\alpha\} = \underline{\underline{T}}^t_{\varphi}\{a_i\}$$

Velocity Vector Components **may be obtained by time derivative of Eq. (3.26) as:**

$$\{\dot{x}_i\} = \dot{\underline{\underline{T}}}_{\varphi}\{x_\alpha\} + \underline{\underline{T}}_{\varphi}\{\dot{x}_\alpha\} \quad \Rightarrow \quad \{\underline{v}_i\} = \dot{\underline{\underline{T}}}_{\varphi}\begin{Bmatrix} R \\ 0 \\ Z \end{Bmatrix} + \underline{\underline{T}}_{\varphi}\begin{Bmatrix} \dot{R} \\ 0 \\ \dot{Z} \end{Bmatrix}$$ **, but:**

$$\{v_\alpha\} = \underline{\underline{T}}^t_{\varphi}\{\underline{v}_i\} = \underline{\underline{T}}^t_{\varphi}\dot{\underline{\underline{T}}}_{\varphi}\begin{Bmatrix} R \\ 0 \\ Z \end{Bmatrix} + \underline{\underline{T}}^t_{\varphi}\underline{\underline{T}}_{\varphi}\begin{Bmatrix} \dot{R} \\ 0 \\ \dot{Z} \end{Bmatrix} = \begin{Bmatrix} \dot{R} \\ R\dot{\varphi} \\ \dot{Z} \end{Bmatrix} = \begin{Bmatrix} v_R \\ v_\varphi \\ v_Z \end{Bmatrix} \qquad \textbf{(3.27)}$$

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Similarly, *Acceleration Vector Components* are obtained as:

$$\{a_i\} = \ddot{\underline{\underline{T}}}_{\varphi}\{x_\alpha\} + 2\dot{\underline{\underline{T}}}_{\varphi}\{\dot{x}_\alpha\} + \underline{\underline{T}}_{\varphi}\{\ddot{x}_\alpha\}$$

$$\{a_\alpha\} = \underline{\underline{T}}^t_{\varphi}\{a_i\} = \underline{\underline{T}}^t_{\varphi}\ddot{\underline{\underline{T}}}_{\varphi}\{x_\alpha\} + 2\underline{\underline{T}}^t_{\varphi}\dot{\underline{\underline{T}}}_{\varphi}\{\dot{x}_\alpha\} + \underline{\underline{T}}^t_{\varphi}\underline{\underline{T}}_{\varphi}\{\ddot{x}_\alpha\}$$

$$\{a_\alpha\} = \underline{\underline{T}}^t_{\varphi}\ddot{\underline{\underline{T}}}_{\varphi}\begin{Bmatrix} R \\ 0 \\ Z \end{Bmatrix} + \begin{Bmatrix} \ddot{R} \\ 2\dot{R}\dot{\varphi} \\ \ddot{Z} \end{Bmatrix} = \begin{Bmatrix} \ddot{R} - R\dot{\varphi}^2 \\ 2\dot{R}\dot{\varphi} + R\ddot{\varphi} \\ \ddot{Z} \end{Bmatrix} = \begin{Bmatrix} a_R \\ a_\varphi \\ a_Z \end{Bmatrix} \qquad \textbf{(3.28)}$$

Other Curvilinear Coordinates: {i.e. *Elliptical or Conical Coordinates* (ζ, η, ξ)}; {ζ: Zeta, η: Eta, ξ: Xi/Zai}

Ex: *Plane Elliptical-Hyperbolic Coordinates* (ζ , η):
Consider the following plane elliptical-hyperbolic coordinate system as:

$$x_1 = R(\zeta + \frac{m}{\zeta})\cos\eta \qquad and \qquad x_2 = R(\zeta - \frac{m}{\zeta})\sin\eta$$

where "m" and "R" are constants. *Note that:*

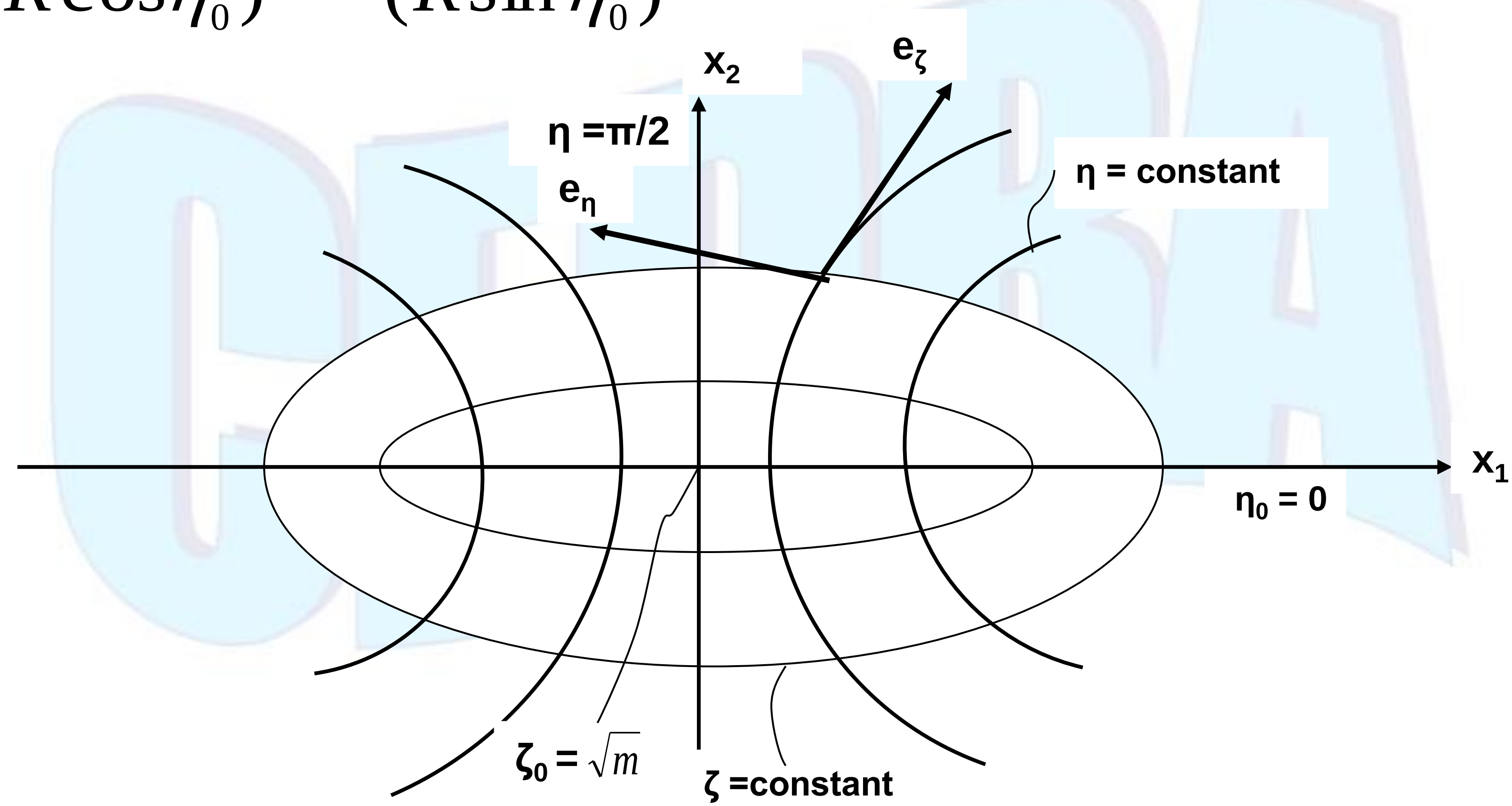
If $\zeta = \zeta_0 =$ *constant*, then the lines of constant ζ are *ellipses* defined by:

$$\frac{x_1^2}{R^2(\zeta_0 + \frac{m}{\zeta_0})^2} + \frac{x_2^2}{R^2(\zeta_0 - \frac{m}{\zeta_0})^2} = 1$$ **(Ellipse in x_1–x_2 plane)**

If η = η_0 = *constant*, then the lines of constant η are *hyperbolas* defined by:

$$\frac{x_1^2}{(R\cos\eta_0)^2} - \frac{x_2^2}{(R\sin\eta_0)^2} = 4m$$

(Hyperbola in x_1–x_2 plane)



Position Vector:

$$\underline{r} = x_1\underline{e}_1 + x_2\underline{e}_2 = R(\zeta + \frac{m}{\zeta})\cos\eta\,\underline{e}_1 + R(\zeta - \frac{m}{\zeta})\sin\eta\,\underline{e}_2$$

Base Vectors:

$$\underline{X}_\zeta = \underline{g}_\zeta = \frac{\partial \underline{r}}{\partial \zeta} = [R(1 + \frac{-m}{\zeta^2})\cos\eta]\underline{e}_1 + [R(1 - \frac{-m}{\zeta^2})\sin\eta]\underline{e}_2$$

$$\underline{X}_\eta = \underline{g}_\eta = \frac{\partial \underline{r}}{\partial \eta} = [-R(\zeta + \frac{m}{\zeta})\sin\eta]\underline{e}_1 + [R(\zeta - \frac{m}{\zeta})\cos\eta]\underline{e}_2$$

(3.29)

Check for Orthogonality:

$$\underline{g}_\zeta \cdot \underline{g}_\eta = [-R^2(\zeta + \frac{m}{\zeta})(1 - \frac{m}{\zeta^2})\sin\eta\cos\eta] + [R^2(\zeta - \frac{m}{\zeta})(1 + \frac{m}{\zeta^2})\sin\eta\cos\eta]$$

$$= -R^2(\zeta - \frac{m^2}{\zeta^3})\sin\eta\cos\eta + R^2(\zeta - \frac{m^2}{\zeta^3})\sin\eta\cos\eta = 0$$

Unit Vectors:

$$\underline{e}_\zeta = \frac{\underline{g}_\zeta}{\left|\underline{g}_\zeta\right|} = \frac{\underline{g}_\zeta}{R\sqrt{[(1 + \frac{m^2}{\zeta^4}) - \frac{2m}{\zeta^2}\cos 2\eta]}}; \qquad \underline{e}_\eta = \frac{\underline{g}_\eta}{\left|\underline{g}_\eta\right|} = \frac{\underline{g}_\eta}{R\zeta\sqrt{[(1 + \frac{m^2}{\zeta^4}) - \frac{2m}{\zeta^2}\cos 2\eta]}}$$

<u>*Orthogonal Transformation*</u> between {x_i} and {x_α}, or {$\underline{e}_i$} and {$\underline{e}_\alpha$}:

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \underline{\underline{T}} \begin{Bmatrix} x_\zeta \\ x_\eta \end{Bmatrix}; \quad where: \quad \underline{\underline{T}} = \frac{\begin{bmatrix} (1-\frac{m}{\zeta^2})\cos\eta & -(1+\frac{m}{\zeta^2})\sin\eta \\ (1+\frac{m}{\zeta^2})\sin\eta & (1-\frac{m}{\zeta^2})\cos\eta \end{bmatrix}}{\sqrt{[(1+\frac{m^2}{\zeta^4}) - \frac{2m}{\zeta^2}\cos 2\eta]}}$$

(3.30)

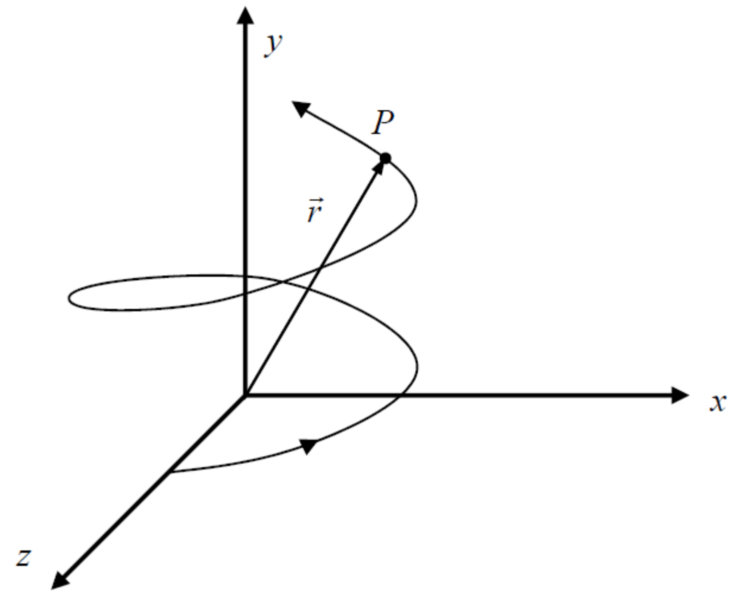
<u>*Special Cases*</u>: If;

$$\zeta_0 = \sqrt{m} \Rightarrow x_1 = R(\sqrt{m} + \frac{m}{\sqrt{m}})\cos\eta = 2R\sqrt{m}\cos\eta, and \quad x_2 = R(\sqrt{m} - \frac{m}{\sqrt{m}})\sin\eta = 0$$

$$\eta_0 = 0 \Rightarrow x_1 = R(\zeta + \frac{m}{\zeta}), \quad and \quad x_2 = 0$$

Example: A particle P shown in the figure moves along a helical path described by the equations:

$$\begin{cases} x = a\cos(\Omega t) \\ y = a\sin(\Omega t) \\ z = bt \end{cases} \quad \textbf{(a)}$$



Determine the velocity and acceleration of the particle P?

***Solution Methods*:**

1. **Rectangular Coordinates; We may write directly from Eq. (a):**

$$\vec{r} = a\cos(\Omega t)\vec{i} + a\sin(\Omega t)\vec{j} + bt\vec{k} \quad \textbf{(b)}$$

$$\vec{v} = \dot{\vec{r}} = -a\Omega\sin(\Omega t)\vec{i} + a\Omega\cos(\Omega t)\vec{j} + b\vec{k} \quad \textbf{(c)}$$

$$\vec{a} = \dot{\vec{v}} = -a\Omega^2\cos(\Omega t)\vec{i} - a\Omega^2\sin(\Omega t)\vec{j} \quad \textbf{(d)}$$

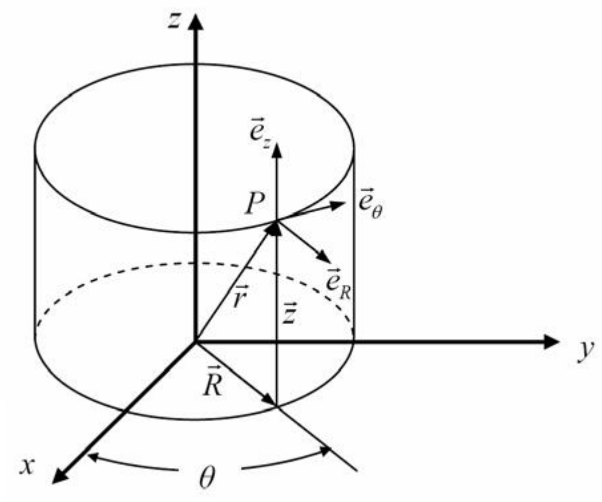
From which:

$$\left|\vec{v}\right| = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} = \sqrt{a^2\,\Omega^2 + b^2} \qquad \textbf{(e)}$$

$$\left|\vec{a}\right| = \sqrt{\ddot{x}^2 + \ddot{y}^2 + \ddot{z}^2} = a\,\Omega^2 \qquad \textbf{(f)}$$

2. **Cylindrical Coordinates; From Eq. (a) according to cylindrical coordinates we can write:**

$$\begin{cases} R = a \;\; ; \quad \dot{R} = \ddot{R} = 0 \\ \theta = \Omega t \;\; ; \quad \dot{\theta} = \Omega \;\; ; \quad \ddot{\theta} = 0 \\ z = bt \;\; ; \quad \dot{z} = b \;\; ; \quad \ddot{z} = 0 \end{cases} \qquad \textbf{(g)}$$



$$\vec{v} = \dot{\vec{R}}\vec{e}_R + R\dot{\theta}\vec{e}_\theta + \dot{z}\vec{e}_z = a\Omega\vec{e}_\theta + b\vec{e}_z \qquad \textbf{(h)}$$

$$\vec{a} = \left(\ddot{R} - r\,\dot{\theta}^2\right)\vec{e}_R + \left(2\,\dot{R}\,\dot{\theta} + R\,\ddot{\theta}\right)\vec{e}_\theta + \ddot{z}\,\vec{e}_z = -a\,\Omega^2\,\vec{e}_R \qquad \textbf{(i)}$$

From which their magnitudes are:

$$\left|\vec{v}\right| = \sqrt{a^2\Omega^2 + b^2} \qquad \text{(j)}$$

$$\left|\vec{a}\right| = a\Omega^2 \qquad \text{(k)}$$

3. Path Variable Coordinates; The velocity is given in path coordinates as:

$$\vec{v} = \dot{s}\,\vec{e}_t \qquad \text{(l)}$$

Directly by use of Eq. (a) we obtain:

$$\begin{cases} \dot{s} = \dfrac{ds}{dt} = \sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2 + \left(\dfrac{dz}{dt}\right)^2} = \sqrt{a^2\Omega^2 + b^2} \\ \ddot{s} = \dfrac{d^2 s}{dt^2} = 0 \end{cases} \qquad \text{(m)}$$

The unit vector $\vec{e}_t$ is defined by:

$$\vec{e}_t = \frac{\vec{v}}{\dot{s}} = \frac{-a\Omega\sin(\Omega t)\vec{i} + a\Omega\cos(\Omega t)\vec{j} + b\vec{k}}{\sqrt{a^2\Omega + b^2}} \qquad \text{(n)}$$

The acceleration is of the form:

$$\vec{a} = \dot{\vec{v}} = \ddot{s}\,\vec{e}_t + \underbrace{\dot{s}\,\dot{\vec{e}}_t} = \ddot{s}\,\vec{e}_t + \underbrace{\frac{\dot{s}^2}{\rho}}\vec{e}_n \qquad \text{(o)}$$

And from Eq. (o) we can write:

$$\dot{\vec{e}}_t = \frac{\dot{s}}{\rho}\vec{e}_n \qquad \text{(p)}$$

Therefore, by differentiating Eq. (n) with respect to time we have:

$$\frac{-a\Omega^2\cos(\Omega t)\vec{i} - a\Omega^2\sin(\Omega t)\vec{j}}{\sqrt{a^2\Omega + b^2}} = \frac{\dot{s}}{\rho}\vec{e}_n \qquad \text{(q)}$$

Considering only the magnitude of both sides of Eq. (q) we have:

$$\frac{\dot{s}}{\rho}=\frac{a\Omega^2}{\sqrt{a^2\Omega+b^2}} \qquad \textbf{(r)}$$

Also from Eqs (q) and (r) we can write:

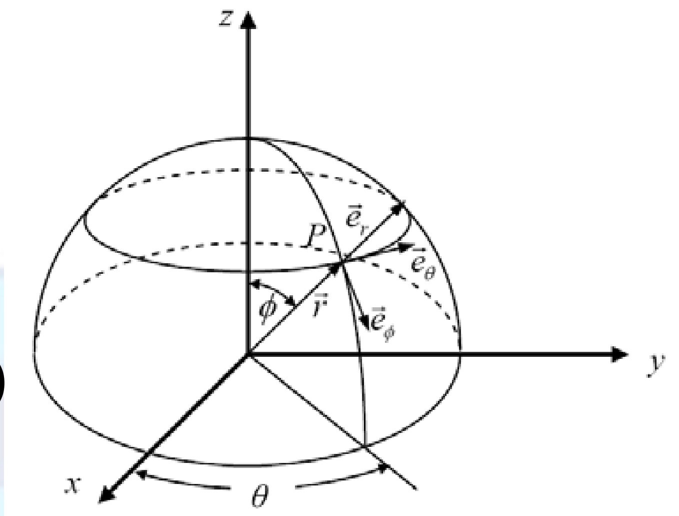
$$\vec{e}_n=-\cos(\Omega t)\vec{i}-\sin(\Omega t)\vec{j} \qquad \textbf{(s)}$$

Substituting these into Eqs. (l) and (o) we obtain:

$$\begin{aligned}\vec{v}&=-a\Omega\sin(\Omega t)\vec{i}+a\Omega\cos(\Omega t)\vec{j}+b\vec{k}\\ \vec{a}&=-a\Omega^2\cos(\Omega t)\vec{i}-a\Omega^2\sin(\Omega t)\vec{j}\end{aligned} \qquad \textbf{(t)}$$

The magnitudes of $\vec{v}$ and $\vec{a}$ agrees with the results obtained by the previous coordinates.

4. **Spherical Coordinates;** **The velocity in Spherical Coordinates is given by:**



$$\vec{v} = \dot{r}\vec{e}_r + r\dot{\phi}\vec{e}_\phi + r\dot{\theta}\sin(\phi)\vec{e}_\theta \qquad \text{(u)}$$

now;

$$|\vec{r}| = \sqrt{x^2 + y^2 + z^2} = \sqrt{a^2 + b^2 t^2} \qquad \text{(v)}$$

and,

$$\tan(\phi) = \frac{a}{bt} \qquad \text{(w)}$$

from which,

$$\sin(\phi) = \frac{a}{\sqrt{a^2 + b^2 t^2}} \quad ; \quad \cos(\phi) = \frac{bt}{\sqrt{a^2 + b^2 t^2}} \qquad \text{(x)}$$

and from Eq. (w) by differentiation we obtain;

$$\dot{\phi} = -\frac{ab}{b^2 t^2}\frac{1}{\sec^2(\phi)} = -\frac{ab}{b^2 t^2}\frac{1}{\left(1 + \tan^2(\phi)\right)} = -\frac{ab}{a^2 + b^2 t^2} \qquad \text{(y)}$$

also,

$$\vec{e}_r = \sin(\phi)\cos(\Omega t)\vec{i} + \sin(\phi)\sin(\Omega t)\vec{j} + \cos(\phi)\vec{k}$$

$$\vec{e}_\phi = \cos(\phi)\cos(\Omega t)\vec{i} + \cos(\phi)\sin(\Omega t)\vec{j} - \sin(\phi)\vec{k} \quad \text{(z)}$$

$$\vec{e}_\theta = -\sin(\Omega t)\vec{i} + \cos(\Omega t)\vec{j}$$

Substituting these results into Eq. (u), we obtain;

$$\vec{v} = \dot{r}\vec{e}_r + r\dot{\phi}\vec{e}_\phi + r\dot{\theta}\sin(\phi)\vec{e}_\theta \quad \text{(u)}$$

$$\vec{v} = \frac{2b^2 t}{2\sqrt{a^2+b^2t^2}}\left(\frac{a}{\sqrt{a^2+b^2t^2}}\cos(\Omega t)\vec{i} + \frac{a}{\sqrt{a^2+b^2t^2}}\sin(\Omega t)\vec{j} + \frac{bt}{\sqrt{a^2+b^2t^2}}\vec{k}\right)$$

$$+\sqrt{a^2+b^2t^2}\,\frac{-ab}{a^2+b^2t^2}\left(\frac{bt}{\sqrt{a^2+b^2t^2}}\cos(\Omega t)\vec{i} + \frac{bt}{\sqrt{a^2+b^2t^2}}\sin(\Omega t)\vec{j} - \frac{a}{\sqrt{a^2+b^2t^2}}\vec{k}\right)$$

$$+\sqrt{a^2+b^2t^2}\,\frac{a\Omega}{\sqrt{a^2+b^2t^2}}\left(-\sin(\Omega t)\vec{i} + \cos(\Omega t)\vec{j}\right)$$

and upon simplification we obtain;

$$\vec{v} = -a\Omega\sin(\Omega t)\vec{i} + a\Omega\cos(\Omega t)\vec{j} + b\vec{k} \qquad \text{(z-1)}$$

Similarly, acceleration in Spherical Coordinate can be computed from:

$$\begin{aligned}\vec{a} = \dot{\vec{v}} = &\left(\ddot{r} - r\dot{\phi}^2 - r\dot{\theta}^2\sin^2(\phi)\right)\vec{e}_r \\ &+ \left(r\ddot{\phi} + 2\dot{r}\dot{\phi} - r\dot{\theta}^2\sin(\phi)\cos(\phi)\right)\vec{e}_\phi \\ &+ \left(r\ddot{\theta}\sin(\phi) + 2\dot{r}\dot{\theta}\sin(\phi) + 2r\dot{\phi}\dot{\theta}\cos(\phi)\right)\vec{e}_\theta\end{aligned} \qquad \text{(z-2)}$$

$$\vec{a} = -a\Omega^2\cos(\Omega t)\vec{i} - a\Omega^2\sin(\Omega t)\vec{j} \qquad \text{(z-3)}$$

Which is the same result as before!!!

It is well clear from this example that a wise selection of coordinates in a given problem will speed up the solution.


متشکرم