بسم الله الرحمن الرحیم

By: Professor Ali Meghdari

Cartesian (Rectangular) Coordinates **$\{x_i\}$:**

Consider a particle traveling in a Cartesian coordinates:

$$\underline{r} = x_1 \underline{e}_1 + x_2 \underline{e}_2 + x_3 \underline{e}_3 = x_i \underline{e}_i$$ **(*Position Vector*)** **(3.10)**

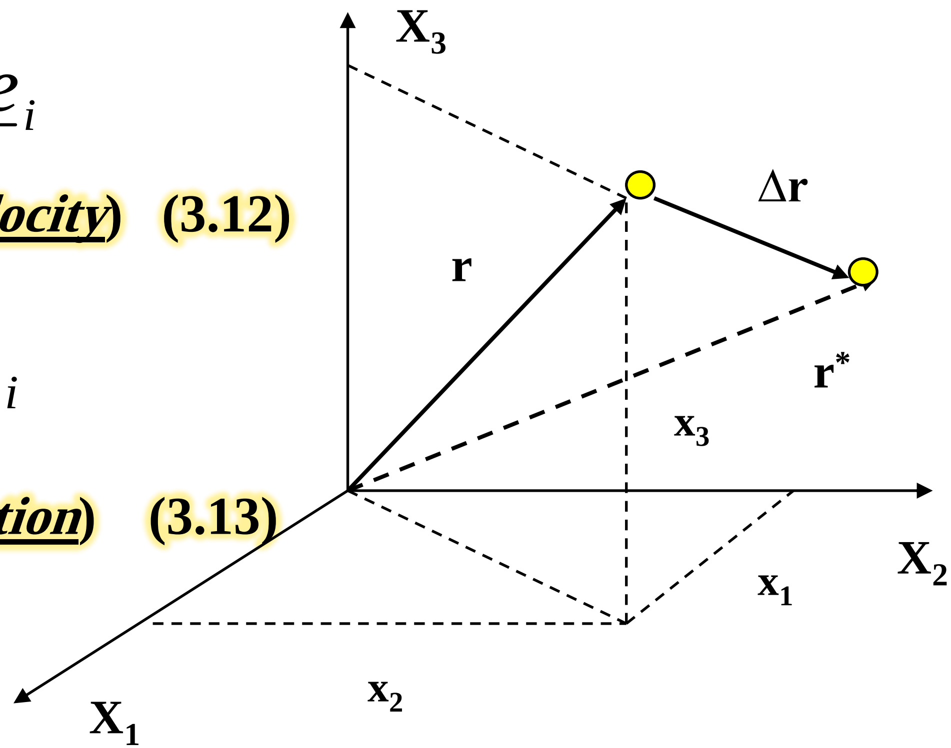
$$\Delta \underline{r} = \underline{r}^* - \underline{r} \equiv \Delta x_i \underline{e}_i = (x_i^* - x_i)\underline{e}_i$$ **(*Particle Displacement*)** **(3.11)**

$$\underline{v} = \frac{d\underline{r}}{dt} = \dot{\underline{r}} \equiv v_i \underline{e}_i = \dot{x}_i \underline{e}_i$$

(*Particle Velocity*) **(3.12)**

$$\underline{a} = \dot{\underline{v}} = \ddot{\underline{r}} \equiv a_i \underline{e}_i = \ddot{x}_i \underline{e}_i$$

(*Particle Acceleration*) **(3.13)**



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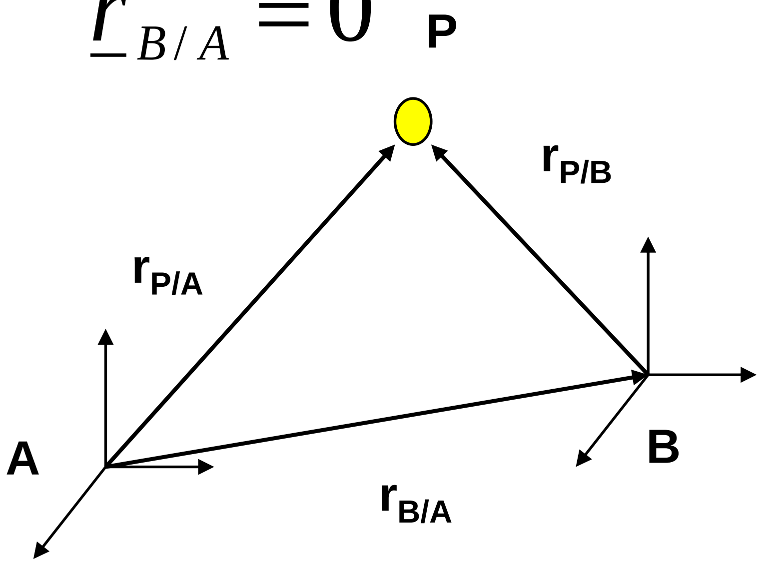
Remark: **If A and B are two different reference points in a single reference frame, then; velocity and acceleration measures would be independent of the reference point.**

$$\underline{r}_{P/A} = \underline{r}_{B/A} + \underline{r}_{P/B} \qquad where \qquad \underline{r}_{B/A} = const.$$

$$\dot{\underline{r}}_{P/A} = \dot{\underline{r}}_{P/B} \qquad where \qquad \dot{\underline{r}}_{B/A} = 0$$

$$\underline{v}_{P/A} = \underline{v}_{P/B} = \underline{v}_{P}$$

$$\underline{a}_{P/A} = \underline{a}_{P/B} = \underline{a}_{P}$$



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Example: Write the position vector of the point "A" in terms of the shown parameter (θ)?

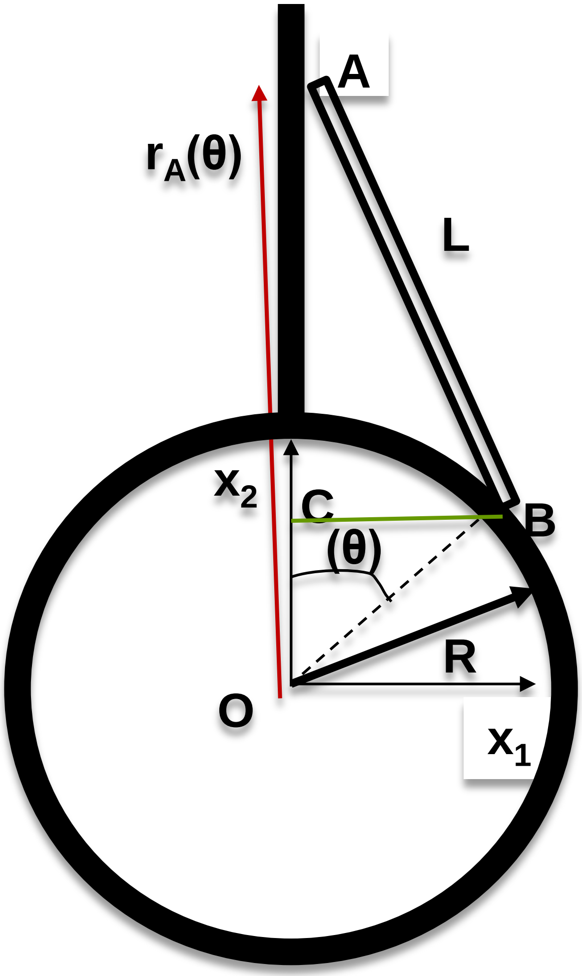
Solution:

$$\overline{BC} = R\sin\theta$$

$$\overline{OC} = R\cos\theta$$

$$\overline{AC} = \sqrt{L^2 - \overline{BC}^2} = R\sqrt{(\frac{L}{R})^2 - \sin^2\theta}$$

$$\underline{r}^A(\theta) = [\overline{OC} + \overline{AC}]\underline{e}_2 = R(\cos\theta + \sqrt{(\frac{L}{R})^2 - \sin^2\theta})\underline{e}_2$$



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Example: An inextensible cord, wrapped around a cylinder, is pulled up as the cylinder rolls without slipping on a level ground. Determine the position, velocity, and acceleration vectors of the tip of the cord in terms of the parameter (θ) and its corresponding derivatives.

Solution:

1. Inextensibility of the cord implies that:

$$\overline{PA} = \overline{P_0A_0} = \frac{\pi}{2}R$$

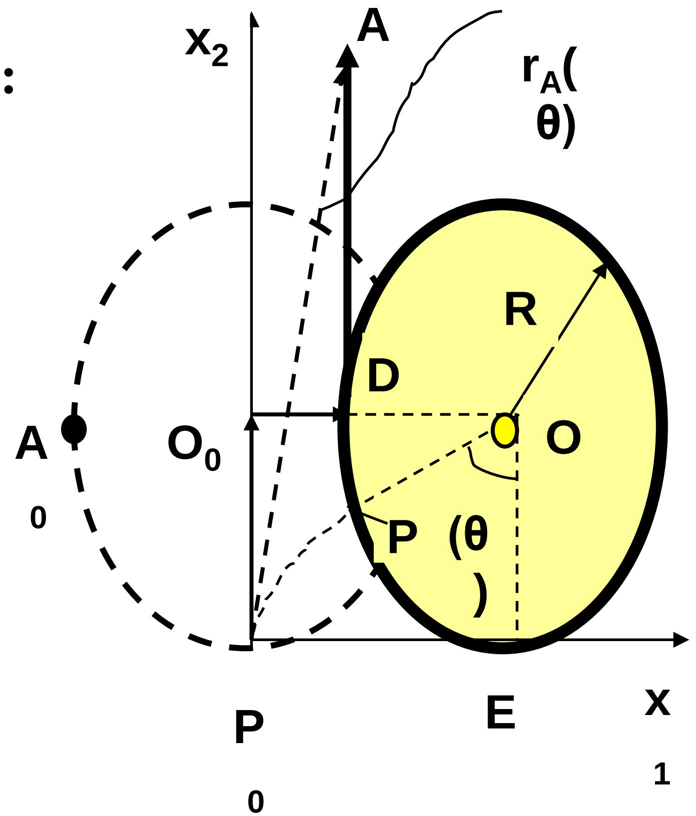
2. Rolling without slipping means that:

$$\overline{P_0E} = \overline{PE} = R\theta$$

3. From 1 and 2, we have:

$$\overline{DA} = \overline{PA} - \overline{PD} = \frac{\pi}{2}R - (\frac{\pi}{2} - \theta)R = R\theta$$

$$\overline{O_0D} = \overline{P_0E} - \overline{OD} = R\theta - R = R(\theta - 1)$$



hence,

$$\underline{r}_A(\theta) = \overline{P_0O_0}\,\underline{e}_2 + \overline{O_0D}\,\underline{e}_1 + \overline{DA}\,\underline{e}_2 = R\underline{e}_2 + R(\theta - 1)\underline{e}_1 + R\theta\underline{e}_2$$

Position vector:

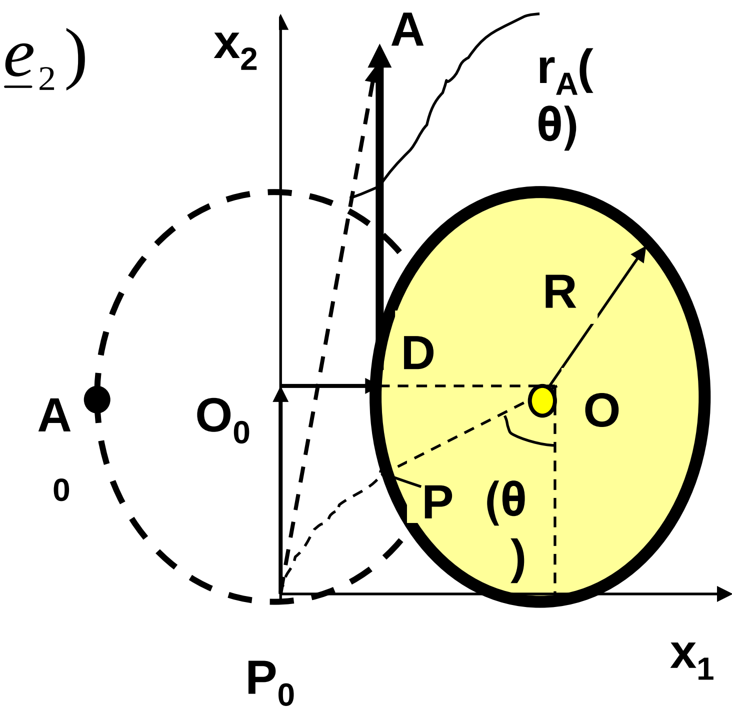
$$\underline{r}_A(\theta) = R(\theta - 1)\underline{e}_1 + R(\theta + 1)\underline{e}_2$$

Velocity vector:

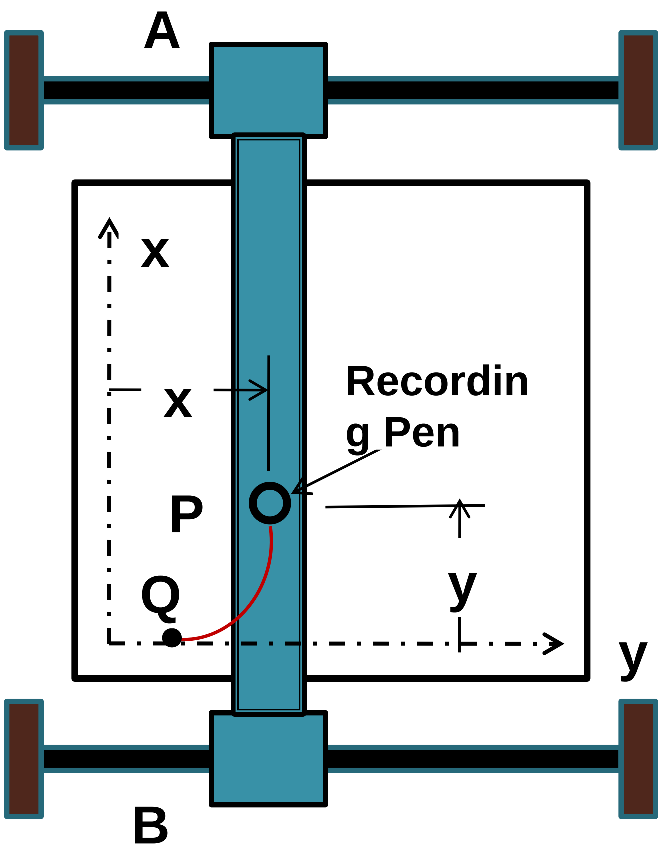
$$\underline{v}_A = \dot{\underline{r}}_A = R\dot{\theta}\underline{e}_1 + R\dot{\theta}\underline{e}_2 = R\dot{\theta}(\underline{e}_1 + \underline{e}_2)$$

Acceleration vector:

$$\underline{a}_A = \dot{\underline{v}}_A = \ddot{\underline{r}}_A = R\ddot{\theta}(\underline{e}_1 + \underline{e}_2)$$



Example: The recording pen is used to draw the line QP on graph paper by an automatic x-y recorder. The velocity of the carriage AB is given as $\dot{x} = 2t + 4$ (ft/s) and the velocity of the pen relative to carriage AB is $\dot{y} = 2/y$ (ft/s). At time t = 0 the pen is at the position (x,y) = (1,0). Determine (a) the equation for the shape of the graph, (b) the velocity and acceleration of point P at t = 2 s, and (c) the slope of the graph at t=2 s.



Solution:

$$\frac{dx}{dt} = 2t + 4 \quad \Rightarrow \quad dx = 2tdt + 4dt \Rightarrow$$

$$x = t^2 + 4t + 1$$

$$\frac{dy}{dt} = 2/y \quad \Rightarrow \quad ydy = 2dt \Rightarrow$$

$$t = y^2/4$$

$$x = \frac{1}{16}y^4 + y^2 + 1 \quad \text{ft}$$

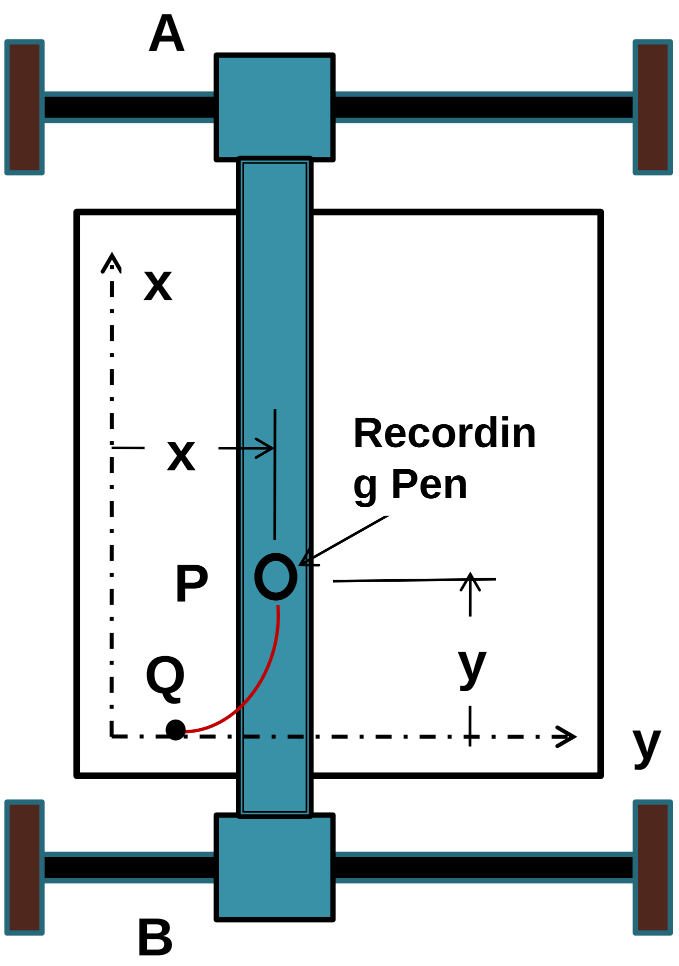
Solution:

$$x = \frac{1}{16} y^4 + y^2 + 1 \quad \text{ft}$$

$$\underline{v} = 8\underline{i} + 0.707\underline{j} \quad \text{ft/s}$$

$$\underline{a} = 2\underline{i} - 0.177\underline{j} \quad \text{ft/s}^2$$

$$\frac{dy}{dx} = \frac{0.707}{8} = 0.0884 \cong 5^\circ$$



***Changing Relativity (Mapping), and Transformations*:**

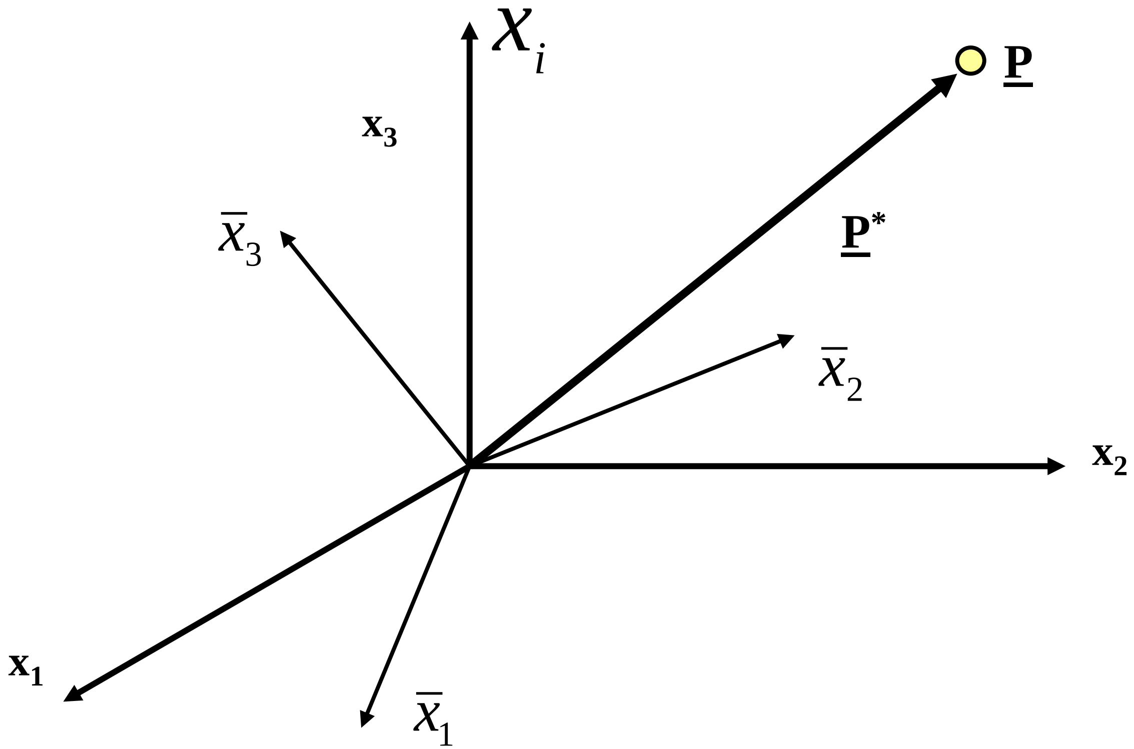
In dynamics, it is sometimes helpful or necessary to describe/ transform components of a vector previously defined in one coordinate frame into the another set of coordinates.

Let us consider a general situation where two coordinate systems, { x_1, x_2, x_3 **} and {** $\bar{x}_1, \bar{x}_2, \bar{x}_3$ **}, are employed to represent the components of a vector.**

(Only orientation of the axes is of interest here, so the origins of the coordinate systems coincide).

Given: Suppose $\underline{P}^*$ is known in the coordinate frame $\{\bar{x}_j\}$,

Find: Define $\underline{P}$ in the coordinate frame { }?



It can be shown that:

$$\underline{P} = \underline{\underline{T}}\,\underline{P}^* \qquad or \qquad \{P_i\}_x = [\ell_{ij}]\{\underline{P}_j^*\}_{\bar{x}}$$

(3.14)

$\underline{\underline{T}} =$

where:

$$\underline{\underline{T}} = [\ell_{ij}] = \begin{bmatrix} l_{1\bar{1}} & l_{1\bar{2}} & l_{1\bar{3}} \\ l_{2\bar{1}} & l_{2\bar{2}} & l_{2\bar{3}} \\ l_{3\bar{1}} & l_{3\bar{2}} & l_{3\bar{3}} \end{bmatrix} \equiv \text{(}\textit{\underline{Matrix of Direction Cosines}}\text{)} \qquad (3.15)$$

$$\ell_{ij} = \cos(x_i, \bar{x}_j), \quad (i.e. \quad l_{1\bar{1}} = \underline{e}_1 \cdot \bar{\underline{e}}_1 = |\underline{e}_1||\bar{\underline{e}}_1| \cos(\underline{e}_1, \bar{\underline{e}}_1) = \cos(x_1, \bar{x}_1) \quad)$$

Columns **of** $\underline{\underline{T}} = [\ell]_{ij}$ **are:**

the projection of the unit vectors of $\bar{X}_j$ **into** X_i

Rows **of** $\underline{\underline{T}} = [\ell]_{ij}$ **are:**

the projection of the unit vectors of X_i **into** $\bar{X}_j$

When the origins coincide, $\underline{\underline{T}}$ **is a** ***Rotation Matrix*****, and** ***Orthogonal*****,**

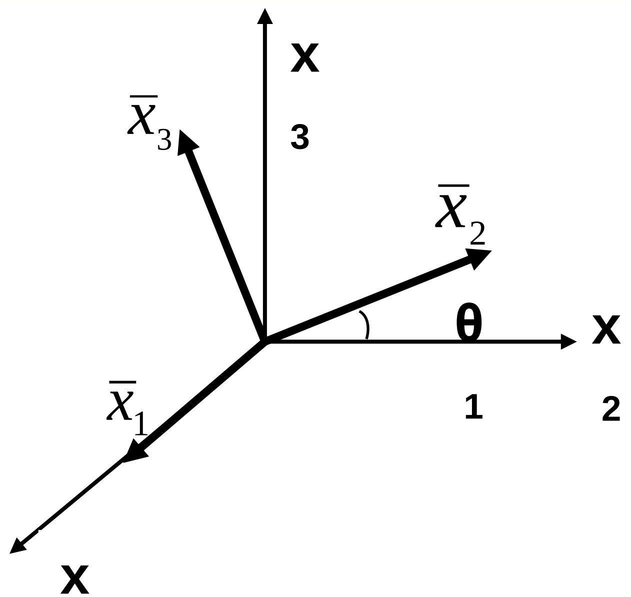
Expressing the relative orientation of frame { $\bar{x}_j$**} with respect to {**X_i **}.**

Inverse of an Orthogonal matrix is equal to its Transpose**:** $(\underline{\underline{T}}^{-1} = \underline{\underline{T}}^{t})$

(*Orthogonal Matrices* are those in which the *dot products* of any of its two columns are equal to *zero*, and the *magnitude* of each one of its columns is equal to *unity*).

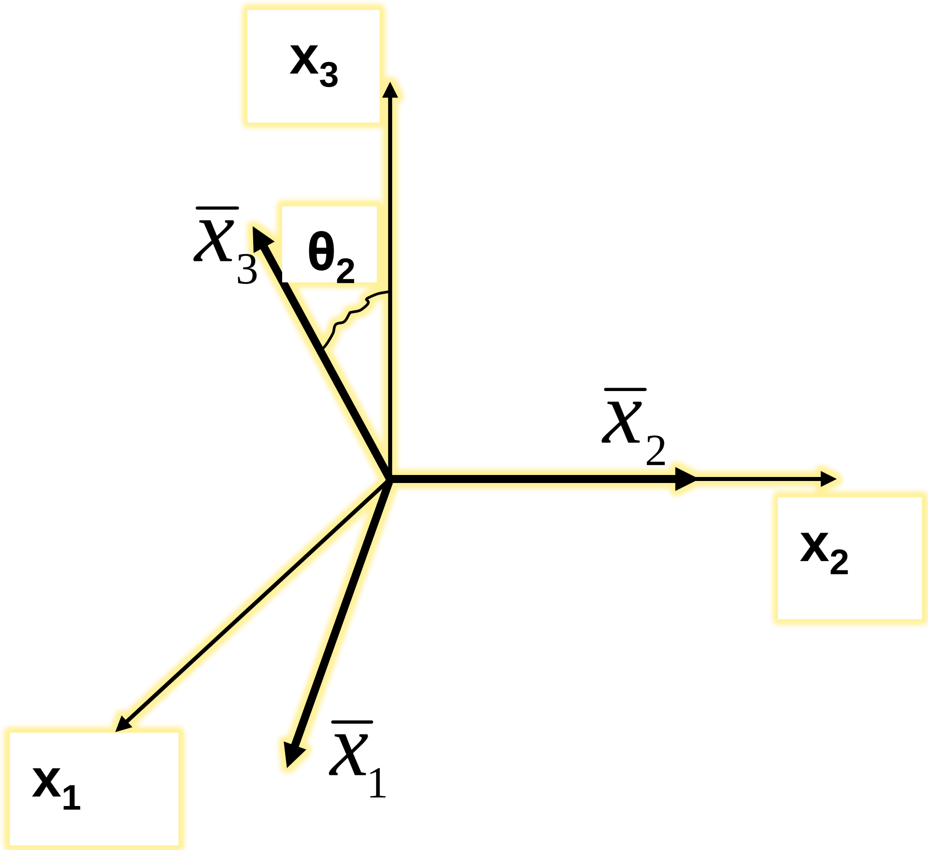
Elementary Rotation Matrices:

Rotation transformations about the coordinate axes are called the *Elementary Rotation Matrices*, and they are defined as follows:

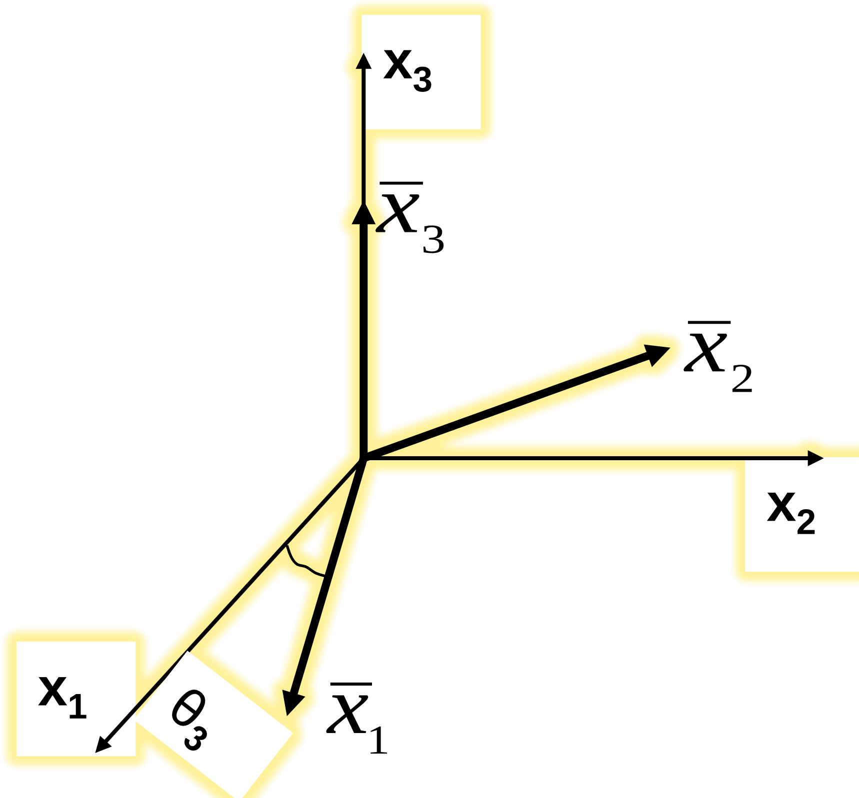


$$\underline{\underline{T}}_1(x_1,\theta_1)=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\theta_1 & -\sin\theta_1\\ 0 & \sin\theta_1 & \cos\theta_1\end{bmatrix} \quad \Rightarrow \underline{x}=\underline{\underline{T}}_1\,\underline{\bar{x}}$$

$$\underline{\underline{T}}_2(x_2,\theta_2)=\begin{bmatrix}\cos\theta_2 & 0 & \sin\theta_2\\ 0 & 1 & 0\\ -\sin\theta_2 & 0 & \cos\theta_2\end{bmatrix} \Rightarrow \underline{x}=\underline{\underline{T}}_2\,\underline{\bar{x}}$$

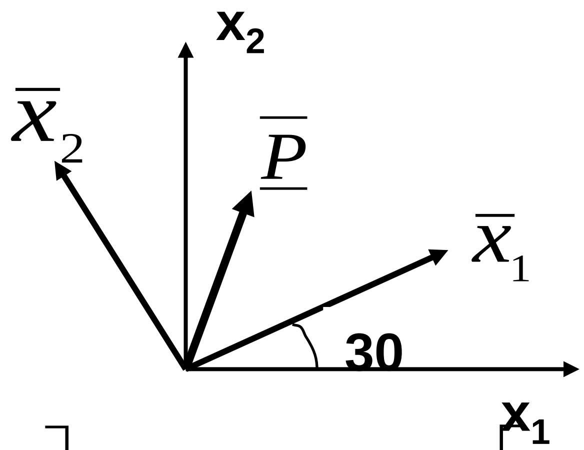


$$\underline{\underline{T}}_3(x_3,\theta_3)=\begin{bmatrix}\cos\theta_3 & -\sin\theta_3 & 0\\ \sin\theta_3 & \cos\theta_3 & 0\\ 0 & 0 & 1\end{bmatrix} \quad \Rightarrow \underline{x}=\underline{\underline{T}}_3\,\overline{x}$$



Ex: Determine the matrix of direction cosines between coordinates $\{x_i\}$ and $\{\bar{x}_j\}$.

***Solution*:**



$$\underline{\underline{T}} = [\ell_{ij}] = \begin{bmatrix} l_{1\bar{1}} & l_{1\bar{2}} \\ l_{2\bar{1}} & l_{2\bar{2}} \end{bmatrix} = [\cos(x_i, \bar{x}_j)] = \begin{bmatrix} \cos(1,\bar{1}) & \cos(1,\bar{2}) \\ \cos(2,\bar{1}) & \cos(2,\bar{2}) \end{bmatrix}$$

$$\underline{\underline{T}} = \begin{bmatrix} \cos 30 & -\sin 30 \\ \sin 30 & \cos 30 \end{bmatrix} = \begin{bmatrix} 0.866 & -0.5 \\ 0.5 & 0.866 \end{bmatrix}$$

Therefore, if :

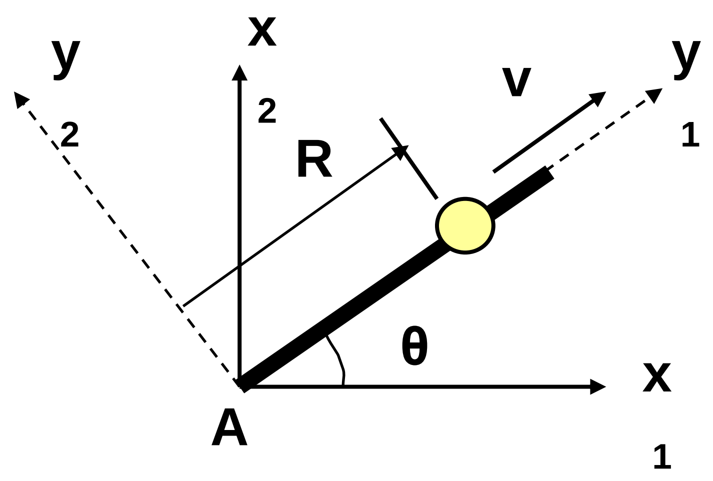
$$\underline{\bar{P}} = 1\underline{\bar{e}}_1 + 1\underline{\bar{e}}_2 \qquad \Rightarrow \underline{P} = \underline{\underline{T}}\,\underline{\bar{P}} = \begin{bmatrix} 0.866 & -0.5 \\ 0.5 & 0.866 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.366 \\ 1.366 \end{bmatrix}$$

Ex: A slender bar lies in the first quadrant of the (x_1-x_2) plane. One of its tips, A, is at the origin. The angular position with respect to x_1-axis is "θ". A bead slides on the slender rod at a distance "R" from A and slides out at a speed "v". Determine the velocity of the bead expressed in frame $\{x_i\}$ for the cases when:

(a)- the rod is fixed?

(b)- the rod spins in the (x_1-x_2) plane about A at an angular velocity $\dot{\theta}$?

Solution:



Let $\{y_j\}$ be another coordinate set such that y_1 coincides with the rod, and y_3 with x_3.

$$\{x_i\} = \underline{\underline{T}}\{y_j\} \qquad where: \underline{\underline{T}} = \begin{bmatrix} C\theta & -S\theta & 0 \\ S\theta & C\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore, position of the bead in $\{x_i\}$ coordinate is:

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{bmatrix} C\theta & -S\theta & 0 \\ S\theta & C\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} R \\ 0 \\ 0 \end{bmatrix} = \begin{Bmatrix} RC\theta \\ RS\theta \\ 0 \end{Bmatrix}$$

(a)- when θ = constant,

$$\Rightarrow \underline{v} = \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{Bmatrix} = \underline{\underline{T}} \begin{bmatrix} v \\ 0 \\ 0 \end{bmatrix} = \begin{Bmatrix} vC\theta \\ vS\theta \\ 0 \end{Bmatrix}$$

(b)- when θ varies with time,

$$\underline{v} = \{v_i\}_x = \{\dot{x}_i\} = \dot{\underline{\underline{T}}} \begin{bmatrix} R \\ 0 \\ 0 \end{bmatrix} + \underline{\underline{T}} \begin{bmatrix} \dot{R} \\ 0 \\ 0 \end{bmatrix}$$

Velocity in $\{x_i\}$ ***coordinate*:**

$$\{\underline{v}\}_x = \{v_i\}_x = \begin{bmatrix} -\dot{\theta}S\theta & -\dot{\theta}C\theta & 0 \\ \dot{\theta}C\theta & -\dot{\theta}S\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} R \\ 0 \\ 0 \end{bmatrix} + \underline{\underline{T}} \begin{bmatrix} v \\ 0 \\ 0 \end{bmatrix} = \begin{Bmatrix} vC\theta - R\dot{\theta}S\theta \\ vS\theta + R\dot{\theta}C\theta \\ 0 \end{Bmatrix}$$

Velocity in $\{y_j\}$ ***coordinate*:**

$$\{\underline{v}\}_y = \{v_j\}_y = \underline{\underline{T}}^t \{v_i\}_x = \begin{bmatrix} C\theta & S\theta & 0 \\ -S\theta & C\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \{v_i\}_x = \begin{Bmatrix} v \\ R\dot{\theta} \\ 0 \end{Bmatrix}$$

ORTHOGONAL CURVILINEAR COORDINATES

Description specifies the position of a point, by giving the value of *3-parameters*, q^α , (i.e. θ, φ, R) which form an *orthogonal mesh* in space.

There exist a unique transformation between the Cartesian Coordinates (x, y, z) and the Orthogonal Coordinates, q^α , (i.e. θ, φ, R), such that:

$$x = x(\theta, \varphi, R), \quad y = y(\theta, \varphi, R), \quad z = z(\theta, \varphi, R) \tag{3.16}$$

$$\theta = \theta(x, y, z), \quad \varphi = \varphi(x, y, z), \quad R=R(x, y, z) \tag{3.17}$$

$x = x(\theta, \varphi, R), \quad y = y(\theta, \varphi, R), \quad z = z(\theta, \varphi, R)$
(3.16)

$\theta = \theta(x, y, z), \quad \varphi = \varphi(x, y, z), \quad R=R(x, y, z)$
(3.17)

When *two* of the parameters of q^{α} (i.e. θ, φ, R) are held *constant* while the *third* is given a range of values, the first group of equations (16) and (17) specifies a curve in space in parametric form.

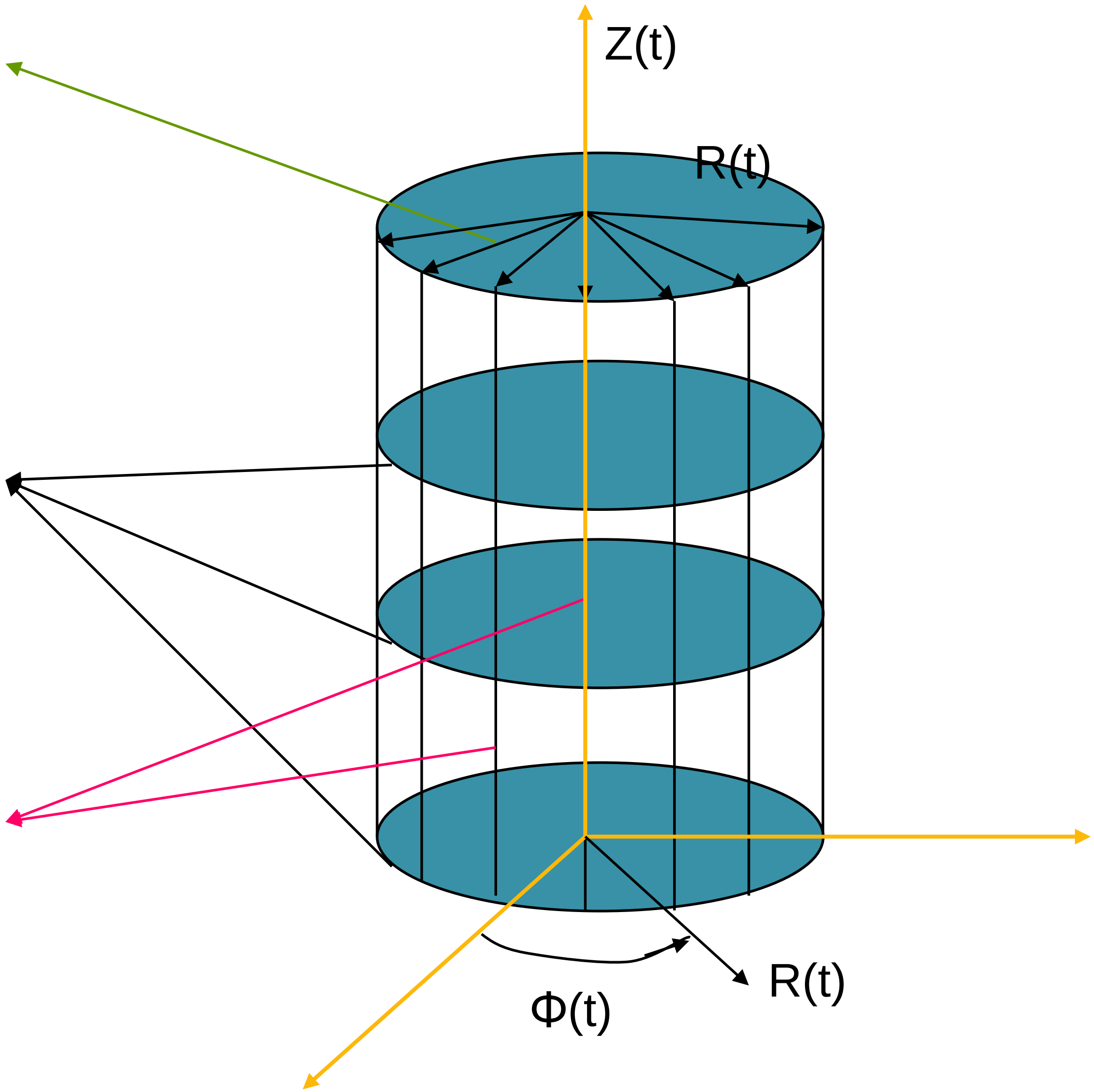
When the constant parameter pair is given a variety of values, the result is *a family of curves*. Repeating this procedure with each of the other pairs of parameters held constant, produces *two more families of curves* (i.e. *called mesh*). The families of curves are mutually *Orthogonal*.

They are named after one of the types of surfaces on which one of the curvilinear coordinates is constant.

$$\begin{cases} X = X(R, \varphi_1, Z_1) \\ X = X(R, \varphi_2, Z_2) \\ \quad . \\ \quad . \\ X = X(R, \varphi_i, Z_i) \end{cases}$$

$$\begin{cases} X = X(R_1, \varphi, Z_1) \\ X = X(R_2, \varphi, Z_2) \\ \quad . \\ \quad . \\ X = X(R_i, \varphi, Z_i) \end{cases}$$

$$\begin{cases} X = X(R_1, \varphi_1, Z) \\ X = X(R_2, \varphi_2, Z) \\ \quad . \\ \quad . \\ X = X(R_i, \varphi_i, Z) \end{cases}$$



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$$\begin{cases} X = X(\theta, \varphi_1, R_1) \\ X = X(\theta, \varphi_2, R_2) \\ \quad . \\ \quad . \\ X = X(\theta, \varphi_i, R_i) \end{cases}$$

$$\begin{cases} X = X(\theta_1, \varphi, R_1) \\ X = X(\theta_2, \varphi, R_2) \\ \quad . \\ \quad . \\ X = X(\theta_i, \varphi, R_i) \end{cases}$$

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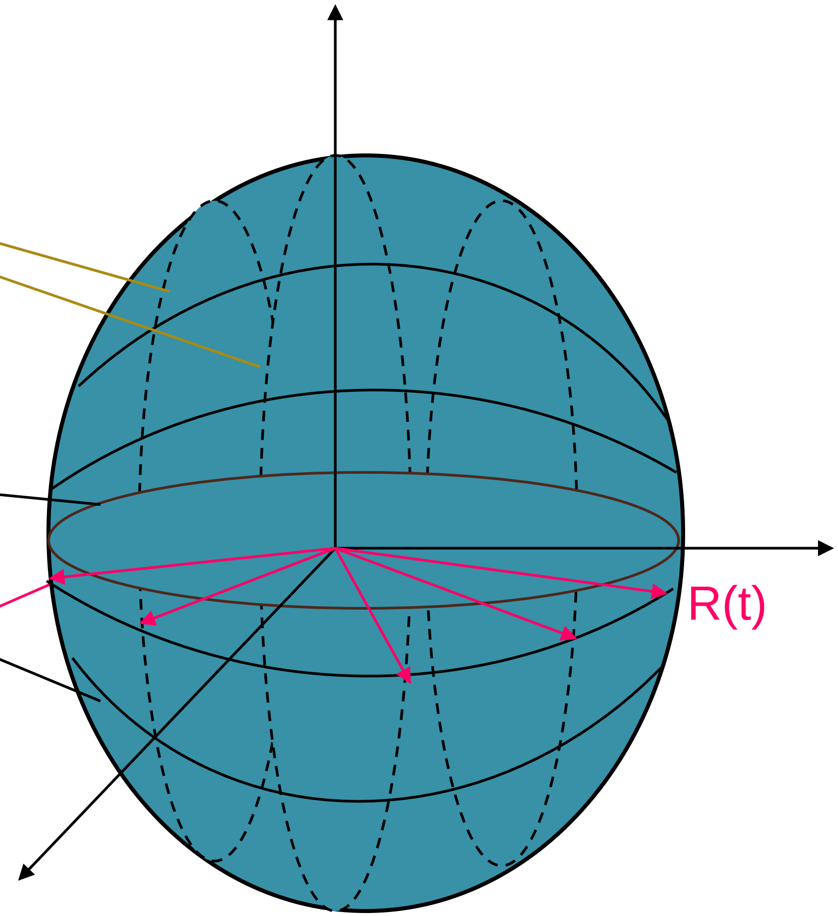


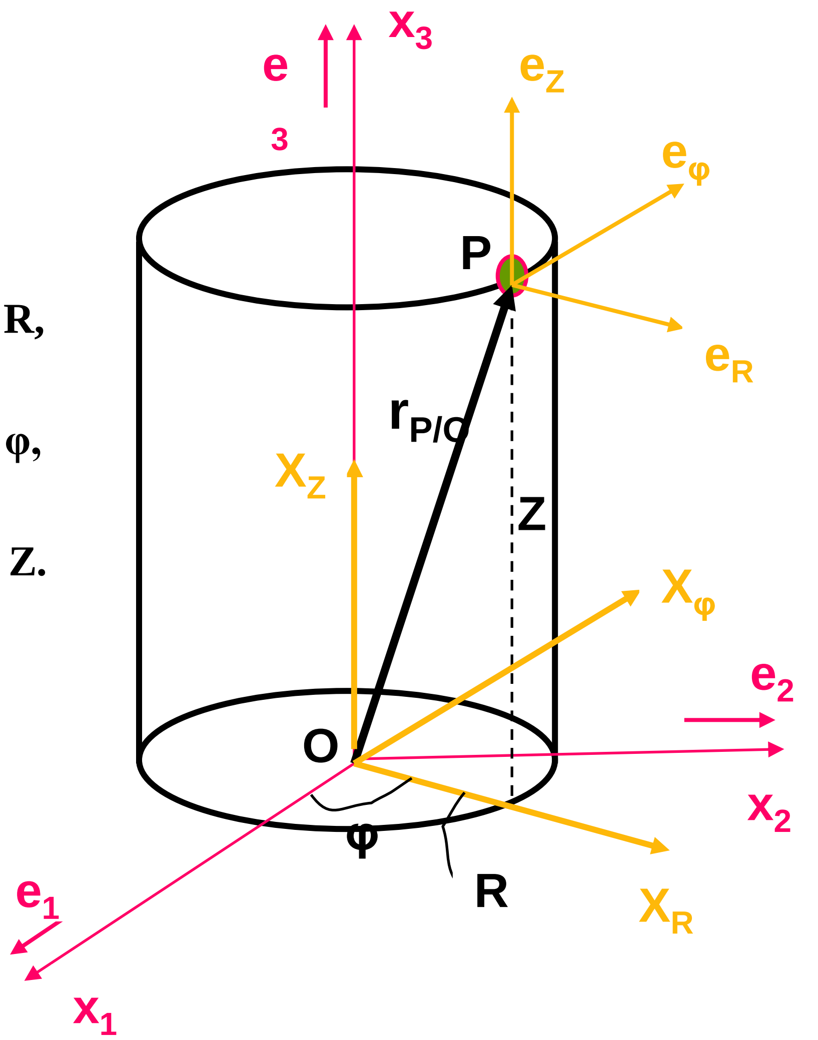
Cylindrical Coordinates (R, φ, Z):

Base Vectors are: {$\underline{x}_R$, $\underline{x}_\varphi$, $\underline{x}_Z$}

$\underline{e}_R$: unit vector in the direction of increasing R,

e_φ : unit vector in the direction of increasing φ,

e_Z : unit vector in the direction of increasing Z.



Position Vector:

$$\underline{r}_{P/O} = R\underline{e}_R + Z\underline{e}_Z \quad \textbf{(3.18)}$$

Velocity Vector:

$$\underline{v}_P = \dot{R}\underline{e}_R + R\underline{\dot{e}}_R + \dot{Z}\underline{e}_Z + Z\underline{\dot{e}}_Z$$

unit vectors {$\underline{e}_R$, $\underline{e}_\varphi$, $\underline{e}_Z$} all rotate with an angular velocity " $\dot{\varphi}$ ",

then using *Jaumann Rate*, we have:

$$\underline{\dot{e}}_R = \underline{\omega}_R \times \underline{e}_R = (\dot{\varphi}\underline{e}_Z) \times (\underline{e}_R) = \dot{\varphi}\underline{e}_\varphi \qquad and \qquad \underline{\dot{e}}_Z = 0$$

$$\underline{v}_P = \dot{R}\underline{e}_R + R\dot{\varphi}\underline{e}_\varphi + \dot{Z}\underline{e}_Z \qquad \textbf{(3.19)}$$

Acceleration Vector:

$$\underline{a}_P = \ddot{R}\underline{e}_R + \dot{R}\underline{\dot{e}}_R + \dot{R}\dot{\varphi}\underline{e}_\varphi + R\ddot{\varphi}\underline{e}_\varphi + R\dot{\varphi}\underline{\dot{e}}_\varphi + \ddot{Z}\underline{e}_Z + \dot{Z}\underline{\dot{e}}_Z$$

$$\underline{\dot{e}}_\varphi = \underline{\omega}_\varphi \times \underline{e}_\varphi = (\dot{\varphi}\underline{e}_Z) \times (\underline{e}_\varphi) = -\dot{\varphi}\underline{e}_R \qquad and \qquad \underline{\dot{e}}_Z = 0$$

$$\underline{a}_P = (\ddot{R} - R\dot{\varphi}^2)\underline{e}_R + (2\dot{R}\dot{\varphi} + R\ddot{\varphi})\underline{e}_\varphi + \ddot{Z}\underline{e}_Z = a_R\underline{e}_R + a_\varphi\underline{e}_\varphi + a_Z\underline{e}_Z \quad \textbf{(3.20)}$$

- a_R : ***Radial Acceleration***
- a_φ : ***Transverse Acceleration***
- a_Z : ***Axial Acceleration***
- $2\dot{R}\dot{\varphi}$: ***Coriolis Acceleration (due*** **to the simultaneous change in "R" and "φ" with respect to time).**

متشکرم