
بسم الله الرحمن الرحيم

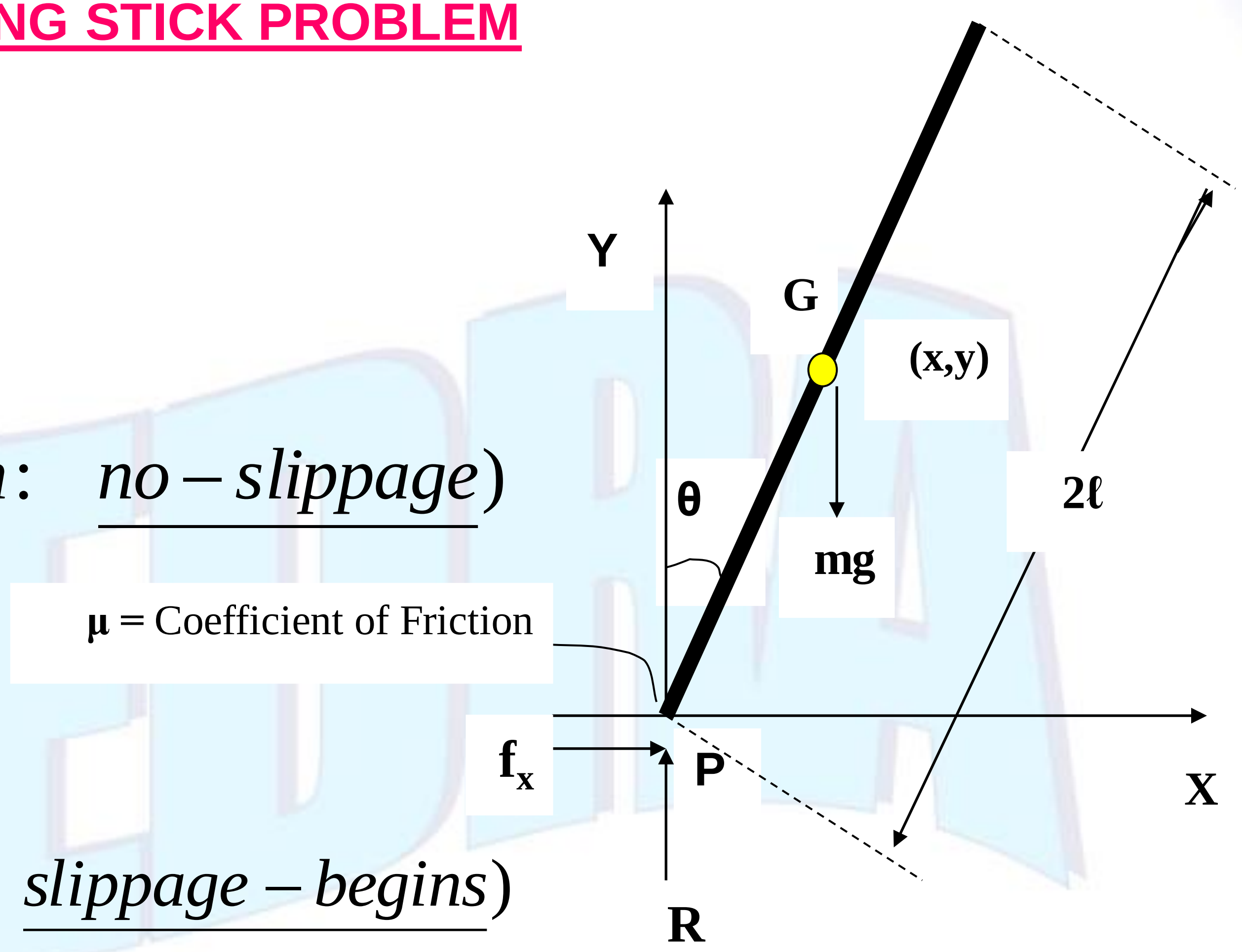
FALLING and SLIPPING

Example: **THE FALLING STICK PROBLEM**

$$f_x = \mu R$$

$$\frac{f_x}{R} < \mu \qquad (when: \quad \underline{no - slippage})$$

$$\frac{f_x}{R} = \mu \qquad (when: \quad \underline{slippage - begins})$$



From Kinematics:

$$x=\ell\sin\theta \quad \Rightarrow \dot{x}=\ell\dot{\theta}\cos\theta \quad \Rightarrow \ddot{x}=\ell\ddot{\theta}\cos\theta-\ell\dot{\theta}^2\sin\theta \qquad \textbf{(1)}$$

$$y=\ell\cos\theta \quad \Rightarrow \dot{y}=-\ell\dot{\theta}\sin\theta \quad \Rightarrow \ddot{y}=-\ell(\ddot{\theta}\sin\theta+\dot{\theta}^2\cos\theta) \qquad \textbf{(2)}$$

From Kinetics:

$$\sum \underline{F}=m\underline{a}$$

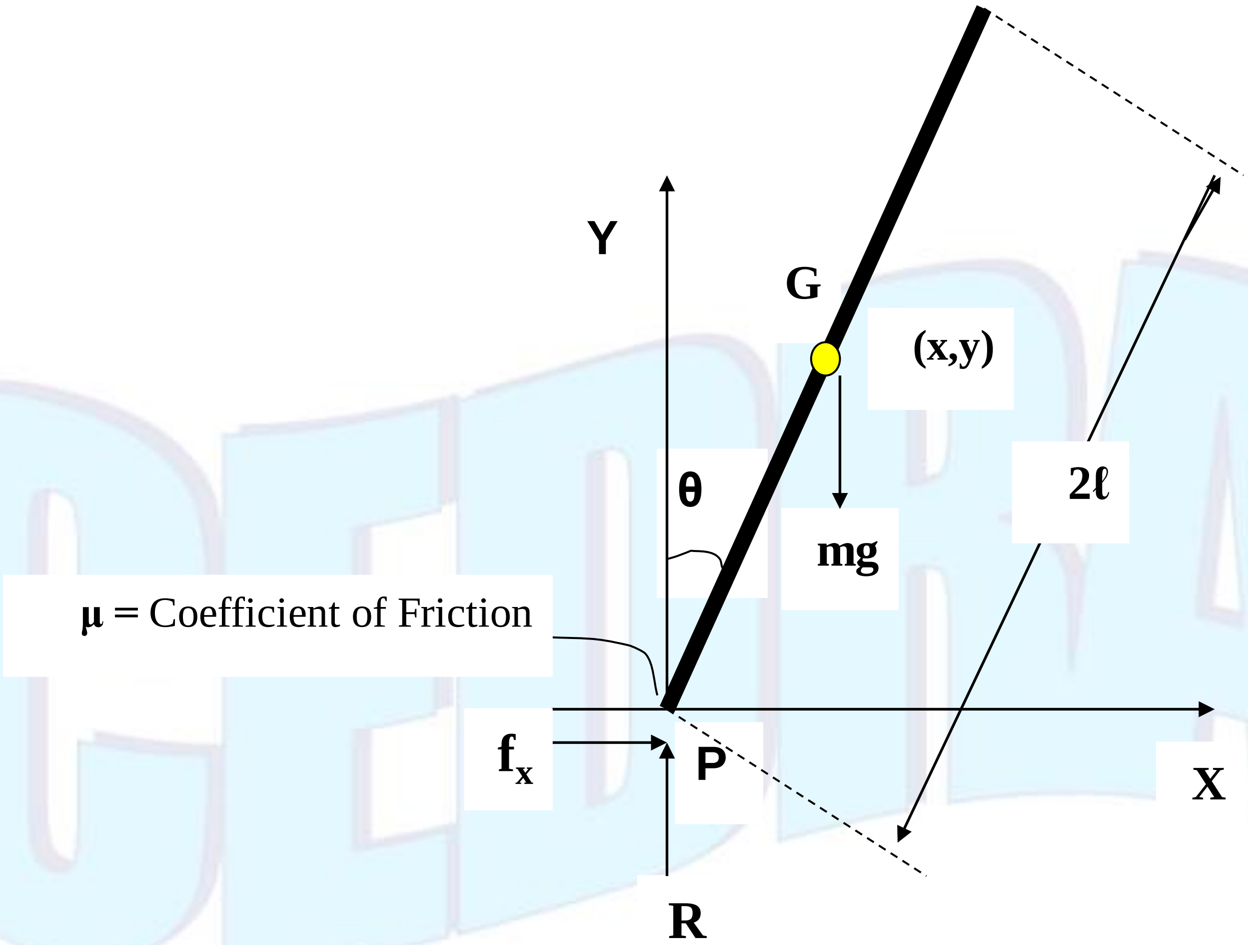
In x-direction:

$$f_x=m\ddot{x}=m\ell(\ddot{\theta}\cos\theta-\dot{\theta}^2\sin\theta) \qquad \textbf{(3)}$$

In y-direction:

$$R-mg=m\ddot{y}=-m\ell(\ddot{\theta}\sin\theta+\dot{\theta}^2\cos\theta) \quad \Rightarrow \qquad \textbf{(4)}$$

$$R=m[g-\ell(\ddot{\theta}\sin\theta+\dot{\theta}^2\cos\theta)]$$


Y
G
(x,y)
θ
mg
2ℓ
μ = Coefficient of Friction
f_x
P
X
R

Let us discuss about the ratio $\dfrac{f_x}{R}$ **:**

First, we will try to eliminate the derivatives of θ (i.e., $\dot{\theta}, \ddot{\theta}$ **) in Equations (3) and (4):**

Applying the *Moment of Momentum* Equation about the point "P", or the mass center "G", we have:

$$M^P = \dot{H}^P = \frac{d}{dt}(I_P \omega) = I_P \ddot{\theta}$$

$$mg\ell \sin\theta = I_P \ddot{\theta} \quad \Rightarrow \quad mk_P^2 \ddot{\theta} = mg\ell \sin\theta \quad \Rightarrow \quad \ddot{\theta} = \frac{g\ell}{k_P^2} \sin\theta \qquad \textbf{(5)}$$

where: $k_P = \sqrt{\dfrac{I_P}{m}} = (\underline{Radius - of - Gyration})$

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where: $$k_P = \sqrt{\frac{I_P}{m}} = (\underline{Radius - of - Gyration})$$

k_P is a measure of length, which defines the mass distribution. Radius of a <u>thin ring</u> having the same mass as the body and with the same moment of inertia about its axis of symmetry as the body has about axis P (It is the effective radius of a body as far as rotation is concerned, as if all the mass is concentrated at k_p).

Equations (3), (4), and (5) form <u>*3-independent*</u> dynamic equations.

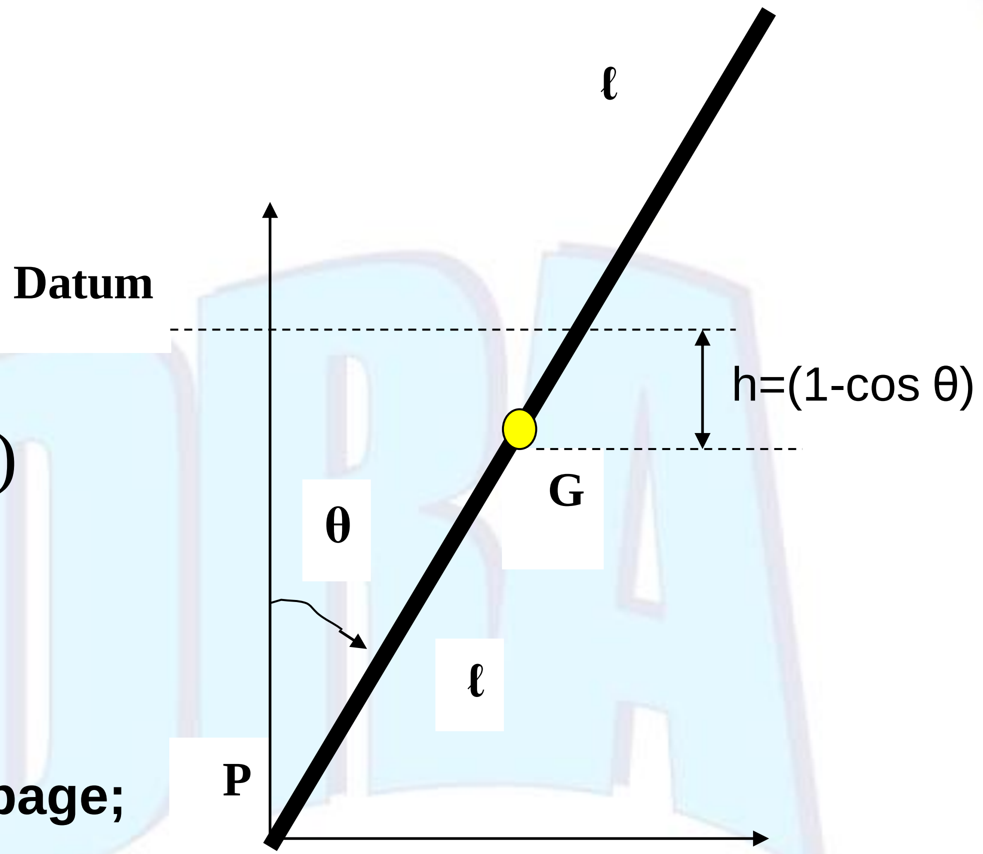
Applying <u>*Energy Equation*</u> (Total Energy is Constant), we have:

T + V = constant

When θ = 0 **(at the top):**

V = P.E. = 0
T = K.E. = 0

Then **:**



V = $-mgh = -mg\ell(1-\cos\theta)$

T = $m\frac{v_G^2}{2} + \frac{I_G\omega^2}{2}$

Or, when P is fixed (before slippage; no dissipation of energy), then:

T = $\frac{1}{2}I_P\omega^2 = \frac{1}{2}I_P\dot{\theta}^2$

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Hence, Energy Equation can be written as (f_x ; is non-working):

T + V = $\frac{I_P\dot{\theta}^2}{2} - mg\ell(1-\cos\theta) = const. = 0 \qquad (by - conventions) \qquad \Rightarrow$

$$\dot{\theta}^2 = \frac{2g\ell}{k_P^2}(1-\cos\theta) \qquad \textbf{(6)}$$

{Integrating Equation (5) with respect to time, would also result in Equation (6)}

Substituting Eqs. (5) and (6) into Eqs. (3) and (4) results in:

$$f_x = \frac{mg\ell^2}{k_P^2}\sin\theta(3\cos\theta - 2) \qquad (7)$$

$$R = mg[1 + \frac{\ell^2}{k_P^2}(3\cos\theta + 1)(\cos\theta - 1)] \qquad (8)$$

$$\frac{f_x}{R} = \frac{3\sin\theta(\cos\theta - \frac{2}{3})}{[(\frac{k_P}{\ell})^2 + 3(\cos\theta - 1)(\cos\theta + \frac{1}{3})]} \qquad (9)$$

☆

This is an expression for $\frac{f_x}{R} = \mu$, as a function of "θ" and ($\frac{k_P}{\ell}$), which is also *independent of mass and g*.

Now, let us analyze the problem:

Consider a homogeneous stick with evenly distributed mass, where;

$$I_P = \frac{1}{3}m(2\ell)^2 = \frac{4}{3}m\ell^2$$

since: $I_P = mk_P^2 \quad \Rightarrow \quad k_P = \frac{2}{\sqrt{3}}\ell = 1.154\ell$

What is the physical meaning of this?

It means that if we take all the mass of the body and put it all at:

$$k_P = 1.154\ell$$

, the rod dynamically would behaves the same as the distributed mass (equivalent to the original form). But if we put all the mass at the mass center, the rod dynamically would not behave the same as the distributed mass.

Note that:

$$k_P = \sqrt{\frac{I_P}{m}} = \sqrt{\frac{I_G + m\ell^2}{m}} = \sqrt{k_G^2 + \ell^2}$$ **(G = mass center)** **(10)**

Also:

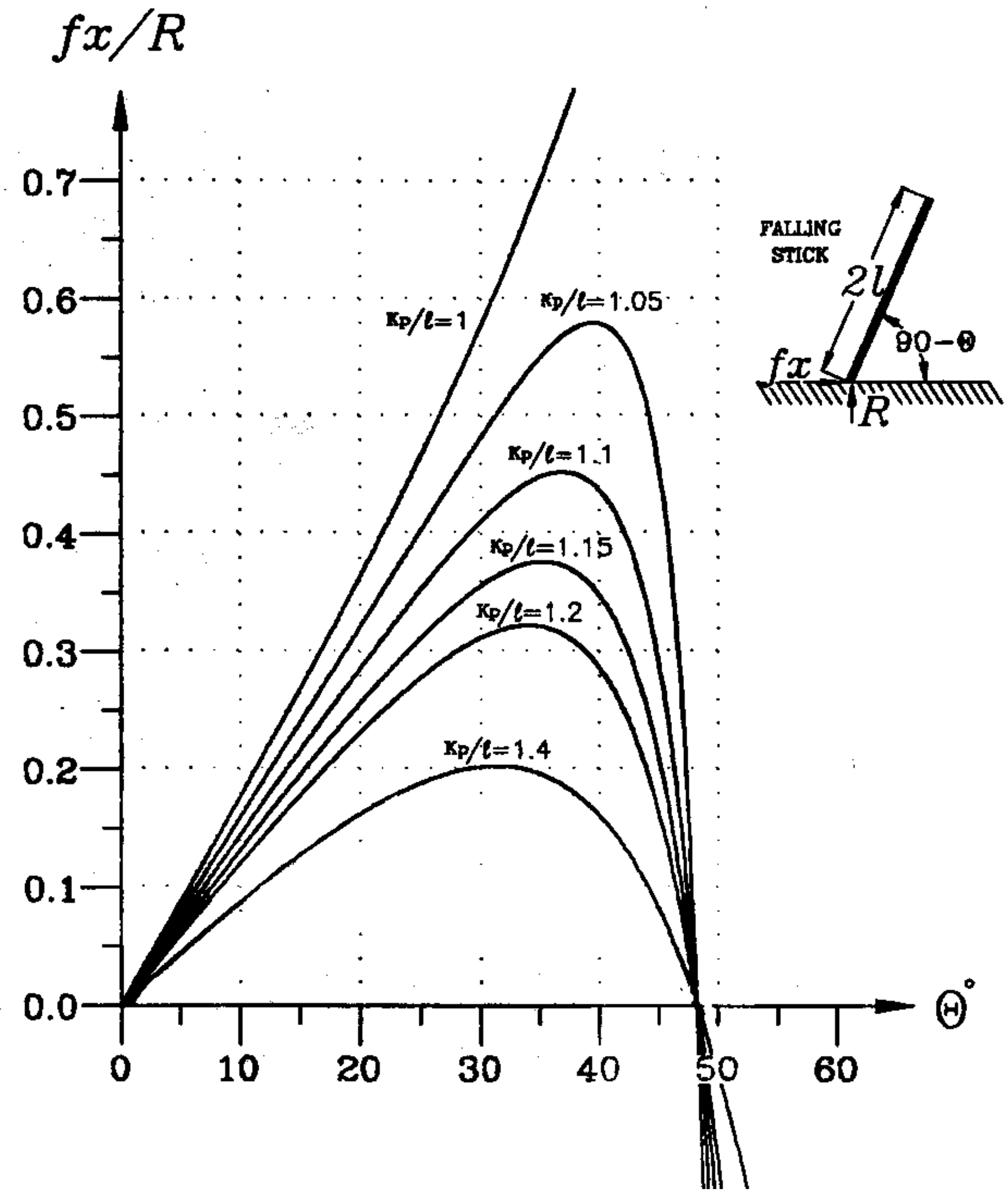
{if all mass is at the mass center G} $0 \le k_G \le \ell$ **{if all mass is stretched to both ends of the stick}**

Therefore, from equation (10), we have:

$$\ell \le k_P \le \sqrt{2}\ell \quad \Rightarrow \quad 1 \le \frac{k_P}{\ell} \le \sqrt{2}$$ ★

Now, if we fix the value of ($1 \le \frac{k_P}{\ell} \le \sqrt{2}$ **) and then plot** $(\frac{f_x}{R})$ **vs. θ , we have:**

(Plot of $\left(\frac{f_x}{R}\right)$vs. θ):



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$$\frac{f_x}{R} = \frac{3\sin\theta(\cos\theta - \frac{2}{3})}{[(\frac{k_P}{\ell})^2 + 3(\cos\theta - 1)(\cos\theta + \frac{1}{3})]} \qquad (9)$$

***Note that*:** **From equation (9), when:**

$$\cos\theta = \frac{2}{3} \quad \Rightarrow \quad \theta = 48.3 \quad \Rightarrow \quad \frac{f_x}{R} = 0$$

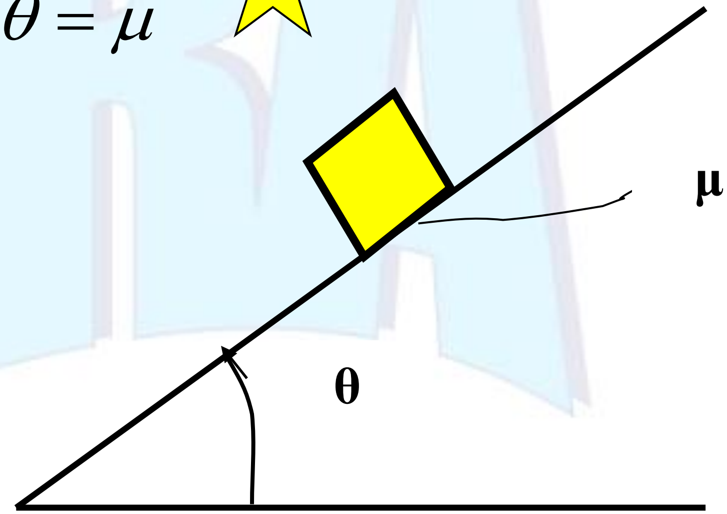
Meaning that *slip starts for sure* when: θ ≤ 48.3

Special Case: consider *a point mass with all the mass at the mass center*, therefore;

$$k_P = \ell \quad \Rightarrow \quad \frac{k_P}{\ell} = 1$$

, and:

$$\frac{f_x}{R} = \frac{3\sin\theta(\cos\theta - \frac{2}{3})}{3\cos\theta(\cos\theta - \frac{2}{3})} = \tan\theta = \mu$$



Similar to the case of a particle on a slope
Where *slippage* begins when: μ=tanθ

PARTICLE KINEMATICS

***Purpose*:**

Define and compute kinematical quantities in various coordinate systems.

***Topics*:**

- **Path Variables Description (*Intrinsic Coordinates*)**
- **Cartesian (*Rectangular*) Coordinates**
- **Orthogonal Curvilinear Coordinates (i.e. Cylindrical, Spherical, Elliptical, etc.)**
- **Coordinate Transformations**

***Expectations*:**

It is expected that all students to be proficient in:

"*Analytical Expression of Kinematical Quantities in Any Given Coordinate Systems*".

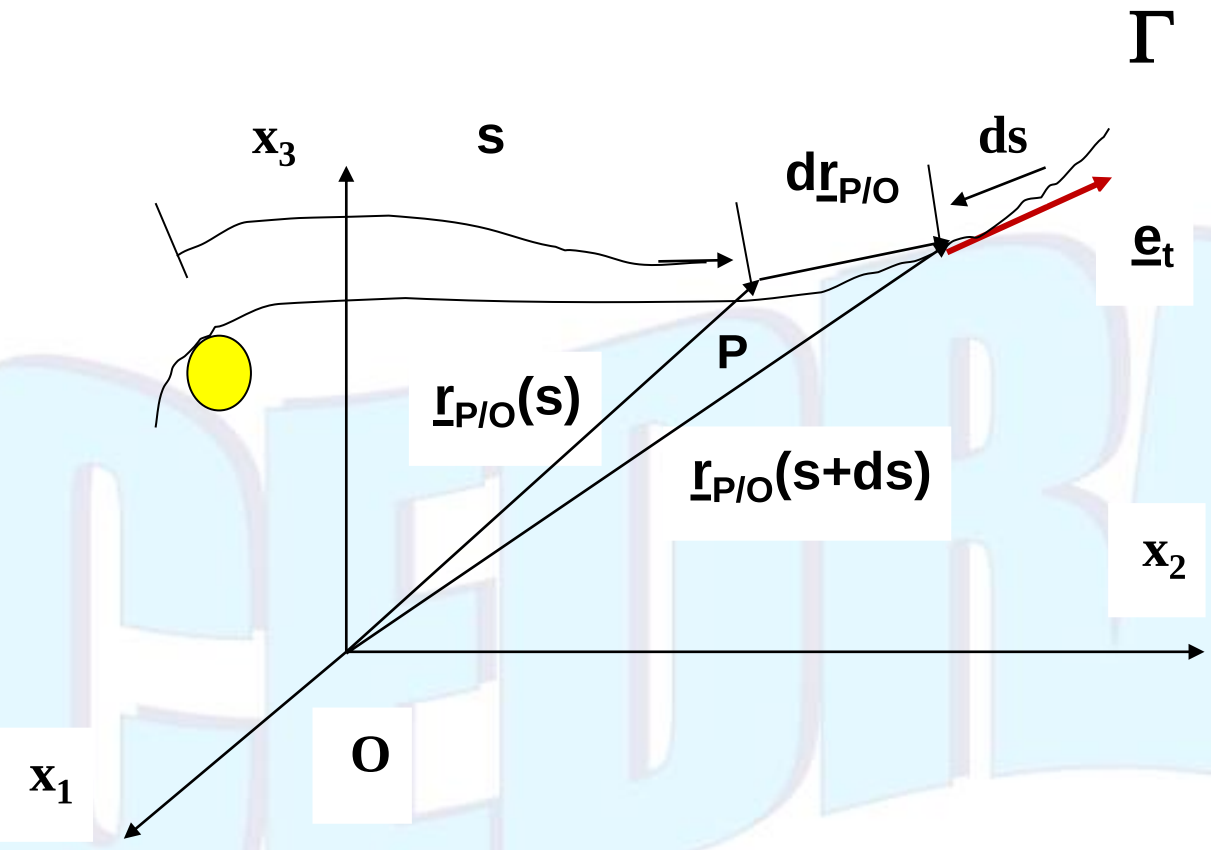
A successful analyst must be able to select and/or to transform to the most appropriate coordinate system that conforms best to the motion.

***Path Variables Description (Intrinsic Coordinates)*:**

Motion of a particle (or a point) is described in terms of the properties of its path (i.e. speedometer & odometer in cars, and road map mileages).

***Intrinsic-Coordinate*; since any change in the basic parameters is associated with the properties of the path (i.e. path dependent).**

Consider a particle traveling through the path " Γ ":



s : arc length

$\underline{r}_{P/O} = \underline{r}_{P/O}(s) =$ *Position Vector at time "t"*, where: $s = s(t)$

Velocity:

$$\underline{v}_P = \frac{d\underline{r}_{P/O}}{dt} = \frac{d\underline{r}_{P/O}}{ds} \cdot \frac{ds}{dt} = \dot{s}\underline{e}_t = v\underline{e}_t \qquad \textbf{(3.1)}$$

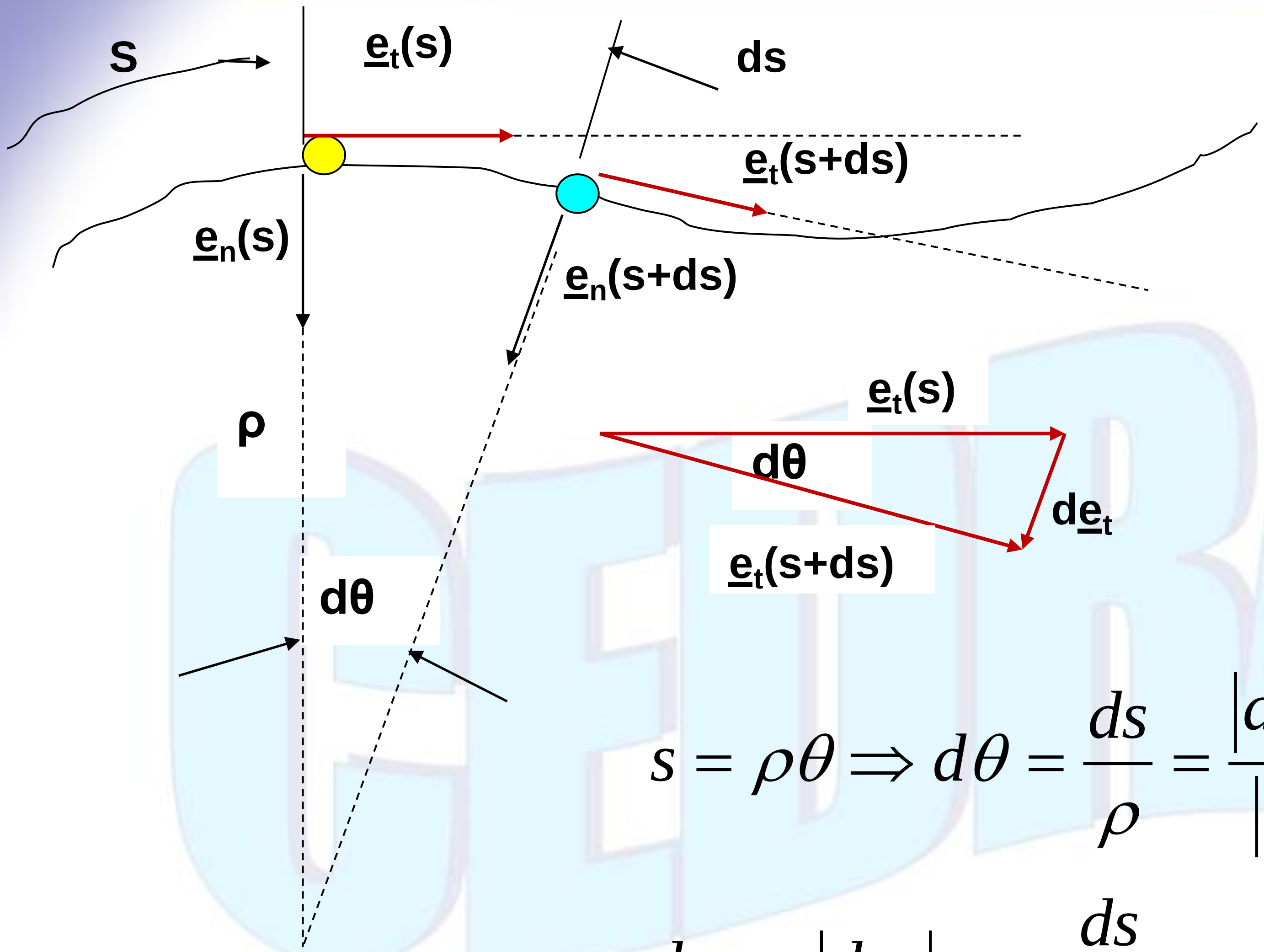
$$where: \underline{e}_t = \frac{d\underline{r}_{P/O}}{ds}; \qquad (\underline{Tangent - to - Path})$$

Accélération: **since $\underline{e}_t = \underline{e}_t(s)$, and s = s(t) , we have :**

$$\underline{a} = \frac{d\underline{v}}{dt} = \frac{d(v\underline{e}_t)}{dt} = \dot{v}\underline{e}_t + v\frac{d\underline{e}_t}{dt}$$

But, $$\frac{d\underline{e}_t}{dt} = \frac{d\underline{e}_t}{ds} \cdot \frac{ds}{dt} \qquad (\underline{Chain - Rule})$$

hence ; $$\underline{a} = \dot{v}\underline{e}_t + v^2\frac{d\underline{e}_t}{ds}$$



$$s = \rho\theta \Rightarrow d\theta = \frac{ds}{\rho} = \frac{|d\underline{e}_t|}{|\underline{e}_t|} = |d\underline{e}_t|$$

$$d\underline{e}_t = |d\underline{e}_t|\underline{e}_n = \frac{ds}{\rho}\underline{e}_n \Rightarrow \frac{d\underline{e}_t}{ds} = \frac{1}{\rho}\underline{e}_n$$

Center of Curvature = C

But,

$$\frac{d\underline{e}_t}{ds} = \frac{1}{\rho}\underline{e}_n \qquad (from - figure)$$

where $\underline{e}_n \perp \underline{e}_t$, and normal to the path.

$$\underline{a} = \dot{v}\underline{e}_t + \frac{v^2}{\rho}\underline{e}_n = a_t\underline{e}_t + a_n\underline{e}_n \qquad \textbf{(3.2)}$$

(*planar motion/path*) ★

a_t **= *tangential acc.*, and**

a_n **= *normal/centripetal acc.***

For a particle traveling on a path or ***a curve in 3-dimension*** (x,y,z) coordinate so that its path is described by the ***position vector*** "$\underline{r}$" as a function of the parameter "t ***within a possible range***", we have:

$$\underline{r} = x(t)\underline{e}_1 + y(t)\underline{e}_2 + z(t)\underline{e}_3 \qquad (3.3)$$

Then, the ***radius of curvature*** "ρ" is computed from the following relation:

$$\frac{1}{\rho} = \frac{1}{(\dot{s})^3}[(\ddot{\underline{r}} \cdot \ddot{\underline{r}})(\dot{s})^2 - (\dot{\underline{r}} \cdot \ddot{\underline{r}})^2]^{1/2} \qquad (3.4)$$

where: *a* ***dot*** denotes differentiation with respect to t, and;

$$\dot{s} = (\dot{\underline{r}} \cdot \dot{\underline{r}})^{1/2} = [(\dot{x})^2 + (\dot{y})^2 + (\dot{z})^2]^{1/2} \qquad (3.5)$$

and,

$$s = \int_{t_0}^{t} [(\dot{x})^2 + (\dot{y})^2 + (\dot{z})^2]^{1/2} dt \equiv (\underline{Arc - Length}) \qquad (3.6)$$

However, for a ***planar curve or path*** like "y=f(x), and z=0, so that t=x, equation (3.4) reduces to:

$$\frac{1}{\rho}=\frac{\left|\frac{d^2y}{dx^2}\right|}{[1+(\frac{dy}{dx})^2]^{3/2}}=\frac{|\ddot{y}|}{[1+(\dot{y})^2]^{3/2}} \qquad \textbf{(3.7)}$$

and, $\dot{s}=(\dot{\underline{r}}\cdot\dot{\underline{r}})^{1/2}=[(\dot{x})^2+(\dot{y})^2]^{1/2}\Rightarrow ds=[(dx)^2+(dy)^2]^{1/2}$

(3.8)

$$s=\int_{x_0}^{x}[1+(\frac{dy}{dx})^2]^{1/2}dx$$

(3.9)

$$\underline{e}_t=\frac{d\underline{r}}{ds}=\frac{d\underline{r}}{dt}\frac{dt}{ds}=\frac{\dot{\underline{r}}}{\dot{s}}$$

$$\underline{e}_n=\rho\frac{d\underline{e}_t}{ds}=\rho\frac{d\underline{e}_t}{dt}\frac{dt}{ds}=\frac{\rho}{(\dot{s})^4}[\ddot{\underline{r}}(\dot{s})^2-\dot{\underline{r}}(\dot{\underline{r}}\cdot\ddot{\underline{r}})]$$

$$\underline{e}_b=\underline{e}_t\times\underline{e}_n=\frac{\rho}{(\dot{s})^3}\dot{\underline{r}}\times\ddot{\underline{r}}\equiv(\underline{binormal-unit-vector})$$


متشکرم