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Introduction to Kane's Dynamics

Advanced Dynamics Course

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Introduction

➢The first step in formulation of motion in a system is to find the mass distribution and kinematics and dynamics parameters in terms of some known variables. The selection of these variables may be considered to be simple but play an important role in the way the final equations and their numerical solution are obtained. The variables that are to describe the configuration of the system are called **Generalized Coordinates (GS).**

➢Lagrange's equations are second order differential equations in the generalized coordinates q_i $(\ i = 1, \ldots, n\ )$.

These may be converted to first-order differential equations or into state-space form in the standard way, by defining an additional set of variables, called motion variables. To convert Lagrange's equations, one defines the motion variables simply as configuration variable derivatives, sometimes called generalized velocities. Then the state vector is made up of the configuration and motion variables: the generalized coordinates and generalized velocities.

Kane’s Method

➢In contrast, Kane’s method motion variables are not generalized velocities, a new definition of a linear function of generalized velocities, known as generalized speeds (GS), replaces them. The total number of independent GS is the same as the degrees of freedom. If the number of constraints is equal to m, then the degrees of freedom will be: $P = n - m$. Selection of GS in terms of generalized velocities (GV) is completely arbitrary, but it affects the amount of calculations and complexity of the problem. One can define the GS as:

$$\{u\}_{P\times1} = [Y]_{P\times n}\{\dot{q}\}_{n\times1} + \{Z\}_{P\times1}$$

In Kane's method, the equations need to determine the linear and angular velocities in terms of $u_1,\dots,u_p$ and $q_1,\dots,q_n$

On the other hand deriving the linear and angular velocities in terms of $\dot{q}_1,\dots,\dot{q}_n$ and $q_1,\dots,q_n$ can be easily done. Therefore, we have to find the relation between $\{\dot{q}\}$ and $\{u\}$

The common form of constraint equations can be written as:

$$[a]_{m\times n}\{\dot{q}\}_{n\times 1}=\{b\}_{m\times 1}$$

According to previous equations, it can be proved that:

$$\{\dot{q}\}_{n\times 1}=[W]_{n\times p}\{u\}_{p\times 1}+\{x\}_{n\times 1}$$

Where

$$\begin{cases}[W]_{n\times P}=A^{-1}B\\ \{x\}=A^{-1}C\end{cases}\qquad A=\begin{bmatrix}[Y]\\ [a]\end{bmatrix}_{n\times n}\qquad B=\begin{bmatrix}[I]_{P\times P}\\ [0]_{m\times P}\end{bmatrix}_{n\times p}\qquad C=\begin{Bmatrix}-\{Z\}\\ \{b\}\end{Bmatrix}_{n\times 1}$$

Reducing the complexity in the derivation of matrix W leads to a simpler procedure in both eventual equations and their solutions.

Consider that the linear and angular velocities are expressed in inertial and body coordinate frame respectively. Since we have obtained $\{\dot{q}\}$ in terms of $\{q\}$ and $\{u\}$, all the velocities can be derived based on these variables.

Now we evaluate the linear and angular momentum for the k^{th} body:

$$\frac{d\{P_k\}}{dt}=\frac{d}{dt}\left(m_k\{V_k\}\right)=m_k\left(\frac{\partial\{V_k\}}{\partial\{u\}}\{\dot{u}\}+\frac{\partial\{V_k\}}{\partial\{q\}}\{\dot{q}\}+\frac{\partial\{V_k\}}{\partial t}\right)$$

$$\frac{d(H_k)}{dt}=[I_k]\{\dot{\omega}_k\}+[\hat{\omega}_k][I_k]\{\omega_k\}$$

$$[\hat{\omega}]=\begin{bmatrix}0 & -\omega_z & \omega_y\\ \omega_z & 0 & -\omega_x\\ -\omega_y & \omega_x & 0\end{bmatrix}$$

$$\frac{d(H_k)}{dt}=[I_k]\left(\frac{\partial\{\omega_k\}}{\partial\{u\}}\{\dot{u}\}+\frac{\partial\{\omega_k\}}{\partial\{q\}}\{\dot{q}\}+\frac{\partial\{\omega_k\}}{\partial t}\right)+[\hat{\omega}_k][I_k]\{\omega_k\}$$

The well known form of Kane's equations is:

$$F_r^*+F_r=0 \qquad \mathrm{r}=1,\ldots,\mathrm{p}$$

Where:

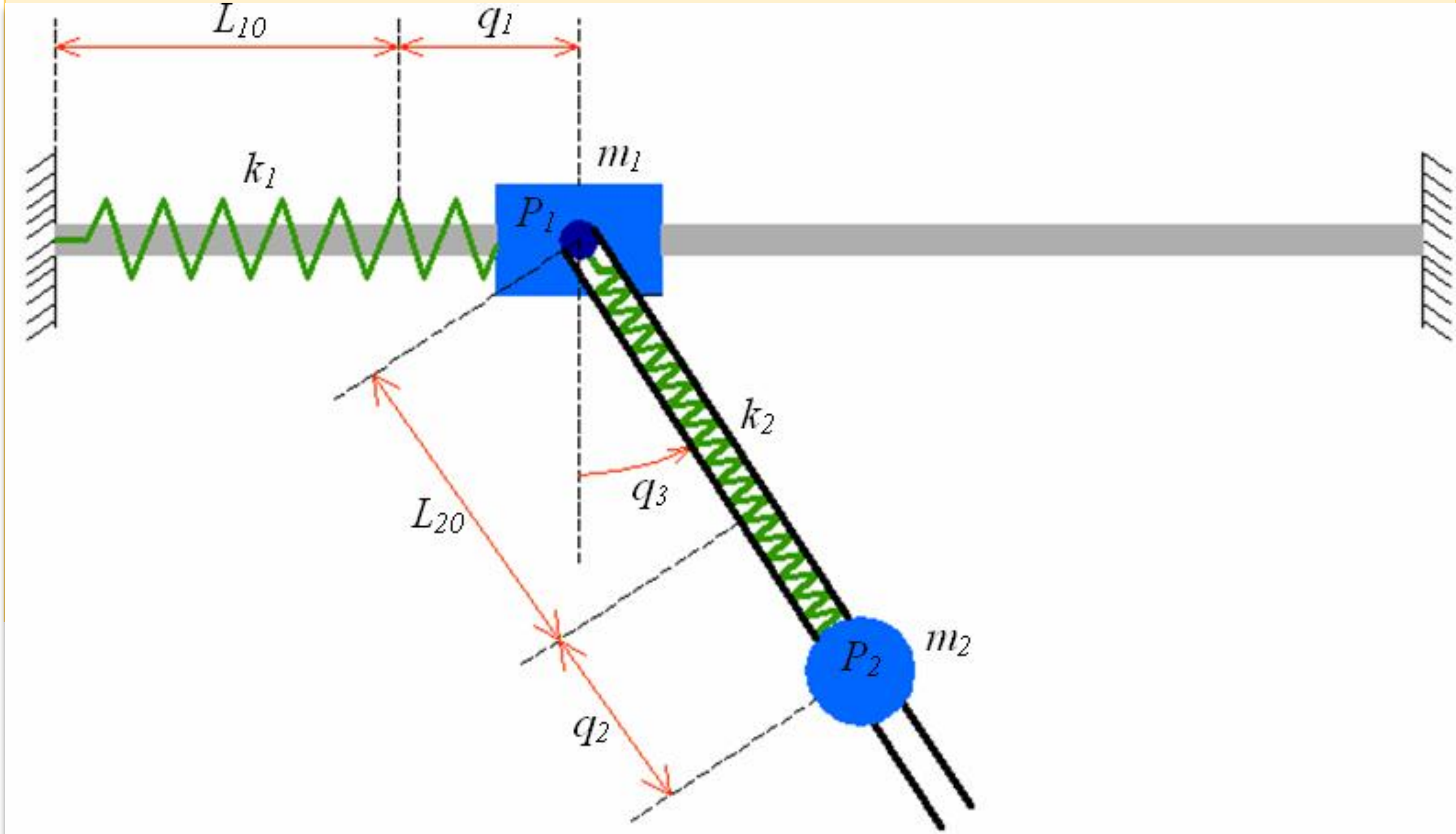
$$F_r^*=-\sum_{k=1}^{\lambda}\left(\frac{d\{P_k\}^T}{dt}\cdot\frac{\partial\{V_k\}}{\partial u_r}+\frac{d\{H_k\}^T}{dt}\cdot\frac{\partial\{\omega_k\}}{\partial u_r}\right)$$

$$F_r=\sum_{i=1}^{\lambda}\vec{V}_r^{\,i}.\vec{R}^{\,i}+\sum_{j=1}^{\gamma}\vec{\omega}_r^{\,j}.\vec{M}^{\,j}$$

Kane's Characteristics

- Ease of Defining Intermediate & State Variables
- The Best Approach for Implementing in Numerical Formulations
- Control Oriented Form of Equations
- Simpler Closed Form Equations for Systems with Complicated Geometries & Nonholomonic Constraints
- Ease of Calculating Internal Forces

➢ **Example 1:** A system without constraint:



Position of bodies as a function of G.C.:

$$\vec{r}^{\,p1} = (L_{10} + q_1)\hat{i}$$

$$\vec{r}^{\,p2} = (L_{10} + q_1 + (L_{20} + q_2)s_3)\hat{i} - (L_{20} + q_2)c_3\hat{j}$$

$$c_3 = \cos(q_3) \qquad s_3 = \sin(q_3)$$

Velocity of bodies as a function of G.C. and G.V.:

$$\vec{v}^{\,p1} = \dot{q}_1\hat{i}$$

$$\vec{v}^{\,p2} = (\dot{q}_1 + \dot{q}_2 s_3 + (L_{20} + q_2)\dot{q}_3 c_3)\hat{i} + (-\dot{q}_2 c_3 + (L_{20} + q_2)\dot{q}_3 s_3)\hat{j}$$

Assume that the selected generalized speeds are as follows:

$$u_r = \dot{q}_r \quad 1 \le r \le 3$$

So the velocity of particles as a function of G.C. and G.S. are:

$$\vec{v}^{\,p1} = u_1 \hat{i}$$

$$\vec{v}^{\,p2} = (u_1 + u_2 s_3 + (L_{20} + q_2) u_3 c_3)\hat{i} + (-u_2 c_3 + (L_{20} + q_2) u_3 s_3)\,\hat{j}$$

By differentiating the accelerations can be determined:

$$\vec{a}^{\,p1} = \dot{u}_1 \hat{i}$$

$$\vec{a}^{\,p2} = \left\{\dot{u}_1 + \dot{u}_2 s_3 + (L_{20} + q_2)\dot{u}_3 c_3 + (L_{20} + q_2) u_3{}^2 s_3\right\}\hat{i} + \left\{-\dot{u}_2 c_3 + (L_{20} + q_2)\dot{u}_3 s_3 + 2u_2 u_3 s_3 + (L_{20} + q_2) u_3{}^2 c_3\right\}\hat{j}$$

The partial velocities are :

$$\vec{v}_1{}^{p1} = \frac{\partial v^{p1}}{\partial u_1} = \hat{i} \qquad \vec{v}_2{}^{p1} = \vec{v}_3{}^{p1} = 0 \qquad \vec{v}_1{}^{p2} = \hat{i}$$

$$\vec{v}_2{}^{p2} = (s_3)\hat{i} + (-c_3)\,\hat{j} \qquad \vec{v}_3{}^{p2} = (L_{20} + q_2)(c_3 \hat{i} + s_3)\,\hat{j}$$

Using Kane's equations, we have:

$$F^*_1 = -(\vec{v}_1^{\,p1}.(m_1\vec{a}^{\,p1}) + \vec{v}_1^{\,p2}.(m_2\vec{a}^{\,p2}))$$

$$F^*_2 = -(\vec{v}_2^{\,p1}.(m_1\vec{a}^{\,p1}) + \vec{v}_2^{\,p2}.(m_2\vec{a}^{\,p2}))$$

$$F^*_3 = -(\vec{v}_3^{\,p1}.(m_1\vec{a}^{\,p1}) + \vec{v}_3^{\,p2}.(m_2\vec{a}^{\,p2}))$$

By substituting from previous equations, we have:

$$F^*_1 = -m_1\dot{u}_1 - m_2\left\{\dot{u}_1 + \dot{u}_2 s_3 + (L_{20}+q_2)\dot{u}_3 c_3 + (L_{20}+q_2)u_3^{\,2} s_3\right\}$$

$$F^*_2 = -m_2\left\{\dot{u}_1 s_3 + \dot{u}_2 + (L_{20}+q_2)u_3^{\,2}(s_3^{\,2} - c_3^{\,2}) - 2u_2 u_3 s_3 c_3\right\}$$

$$F^*_3 = -m_2(L_{20}+q_2)\left\{\dot{u}_1 c_3 + (L_{20}+q_2)\dot{u}_3 + 2(L_{20}+q_2)u_3^{\,2} s_3 c_3 + 2u_2 u_3 s_3^{\,2}\right\}$$

Generalized forces in Kane's method can be obtained via following formulas:

$$F_1 = \vec{v}_1^{\,p1}.\vec{R}^{k1} + \vec{v}_1^{\,p2}.\vec{R}^{k2} + \vec{v}_1^{\,p2}.\vec{R}^{g}$$

$$F_2 = \vec{v}_2^{\,p1}.\vec{R}^{k1} + \vec{v}_2^{\,p2}.\vec{R}^{k2} + \vec{v}_2^{\,p2}.\vec{R}^{g}$$

$$F_3 = \vec{v}_3^{\,p1}.\vec{R}^{k1} + \vec{v}_3^{\,p2}.\vec{R}^{k2} + \vec{v}_3^{\,p2}.\vec{R}^{g}$$

On the other hand, active forces are:

$$\vec{R}^{k1} = -k_1 q_1 \hat{i} \qquad \vec{R}^{k2} = -k_2 q_2 (s_3 \hat{i} - c_3 \hat{j}) \qquad \vec{R}^{g} = -m_2 g \hat{j}$$

So generalized forces are:

$$F_2 = -k_2 q_2 + m_2 g c_3 \qquad F_1 = -k_1 q_1 - k_2 q_2 s_3 \qquad F_2 = -m_2 g s_3$$

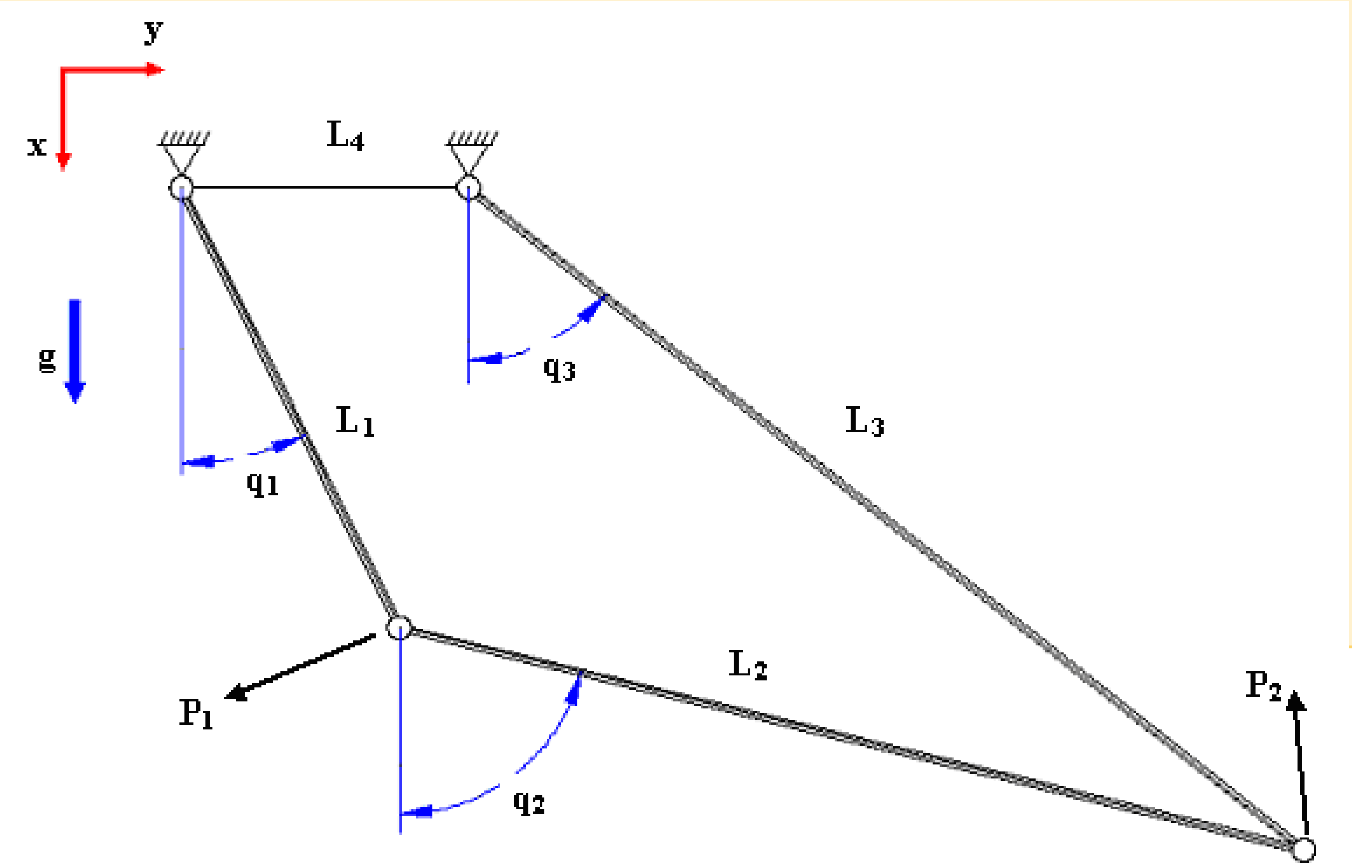
Finally with substituting generalized forces in Kane's equations of motion, we have:

$$m_1\dot{u}_1 + m_2\left\{\dot{u}_1 + \dot{u}_2 s_3 + (L_{20} + q_2)\dot{u}_3 c_3 + (L_{20} + q_2)u_3{}^2 s_3\right\} + k_1 q_1 + k_2 q_2 s_3 = 0$$

$$m_2\left\{\dot{u}_1 s_3 + \dot{u}_2 + (L_{20} + q_2)u_3{}^2(s_3{}^2 - c_3{}^2) - 2u_2 u_3 s_3 c_3\right\} + k_2 q_2 - m_2 g c_3 = 0$$

$$m_2(L_{20} + q_2)\left\{\dot{u}_1 c_3 + (L_{20} + q_2)\dot{u}_3 + 2(L_{20} + q_2)u_3{}^2 s_3 c_3 + 2u_2 u_3 s_3{}^2\right\} + m_2 g s_3 = 0$$

- **Example 2:** This example is a 4 bar mechanism with one degree of freedom. The positive angular direction is counter clockwise. This is a holonomic system with complicated constraints.



The selected G.C. for this problem are: q_1 , q_2 , q_3

Two holonomic constraints of problem are:

$$L_1 c_1 + L_2 c_2 - L_3 c_3 = 0 \qquad L_1 s_1 + L_2 s_2 - L_3 s_3 - L_4 = 0$$

We choose our generalized speed as following:

$$u_1 = \dot{q}_1$$

By differentiating of constraints, we have:

$$L_1 s_1 \dot{q}_1 + L_2 s_2 \dot{q}_2 - L_3 s_3 \dot{q}_3 = 0$$

$$L_1 c_1 \dot{q}_1 + L_2 c_2 \dot{q}_2 - L_3 c_3 \dot{q}_3 = 0$$

By differentiating of constraints, we have:

$$\dot{q}=\begin{bmatrix}\dot{q}_1\\ \dot{q}_2\\ \dot{q}_3\end{bmatrix}=\begin{bmatrix}1\\ \dfrac{L_1\sin(q_3-q_1)}{L_2\sin(q_3-q_2)}\\ \dfrac{L_1\sin(q_2-q_1)}{L_3\sin(q_3-q_2)}\end{bmatrix}u_1=\begin{bmatrix}1\\ A(q)\\ B(q)\end{bmatrix}u_1=Y(q)u_1$$

In the next stage we want to determine the generalized inertia forces. Position and velocity of masses are:

$$\vec{r}^{\,p1}=L_1(c_1\hat{i}+s_1\hat{j}) \qquad \vec{v}^{\,p1}=L_1\dot{q}_1(-s_1\hat{i}+c_1\hat{j})$$

$$\vec{r}^{\,p2}=L_3(c_3\hat{i}+s_3\hat{j})+L_4\hat{j} \qquad \vec{v}^{\,p2}=L_3\dot{q}_3(-s_3\hat{i}+c_3\hat{j})$$

So we have:

$$\vec{v}^{\,p1} = L_1 u_1(-s_1\hat{i} + c_1\hat{j}) \qquad \vec{v}^{\,p2} = L_3 B(q) u_1(-s_3\hat{i} + c_3\hat{j})$$

The partial velocities are as following:

$$\vec{v}_1^{\,p1} = L_1(-s_1\hat{i} + c_1\hat{j}) \qquad \vec{v}_1^{\,p2} = L_3 B(q)(-s_3\hat{i} + c_3\hat{j})$$

The bodies accelerations are as following:

$$\vec{a}^{\,p1} = L_1(\dot{u}_1(-s_1\hat{i} + c_1\hat{j}) + u_1^{\,2}(-c_1\hat{i} - s_1\hat{j}))$$

$$\vec{a}^{\,p2} = L_3(B(q)\dot{u}_1(-s_3\hat{i} + c_3\hat{j}) + B(q)^2 u_1^{\,2}(-c_3\hat{i} - s_3\hat{j}) + \frac{\partial B(q)}{\partial q}\dot{q}u_1(-s_3\hat{i} + c_3\hat{j}))$$

The generalized inertia forces can be obtained as follows:

$$F^{*\,p1}{}_1 = -\vec{v}_1^{\,p1}.(m_1\vec{a}^{\,p1}) = -m_1 L_1{}^2 \dot{u}_1$$

$$F^{*\,p2}{}_1 = -\vec{v}_1^{\,p2}.(m_2\vec{a}^{\,p2}) = -m_2 L_3{}^2 B(q)(B(q)\dot{u}_1 + \frac{\partial B(q)}{\partial q} Y(q) u_1{}^2)$$

$$F^{*}{}_1 = F^{*\,p2}{}_1 + F^{*\,p2}{}_1 = -(m_1 L_1{}^2 \dot{u}_1 + m_2 L_3{}^2 B(q)(B(q)\dot{u}_1 + \frac{\partial B(q)}{\partial q} Y(q) u_1{}^2))$$

On the other hand, the generalized forces can be obtained as follows:

$$F^{W}{}_1 = \vec{v}_1^{\,p1}.\vec{R}^{W1} + \vec{v}_1^{\,p2}.\vec{R}^{W2} = -g(m_1 L_1 s_1 + m_3 L_3 s_3 B(q))$$

$$F^{\tau}{}_1 = \vec{\omega}_1^{\,\tau}.\vec{R}^{\tau} = \tau$$

$$F_1 = F^{\tau}{}_1 + F^{W}{}_1 = \tau - g(m_1 L_1 s_1 + m_3 L_3 s_3 B(q))$$

Finally the only equation of motion is as follows:

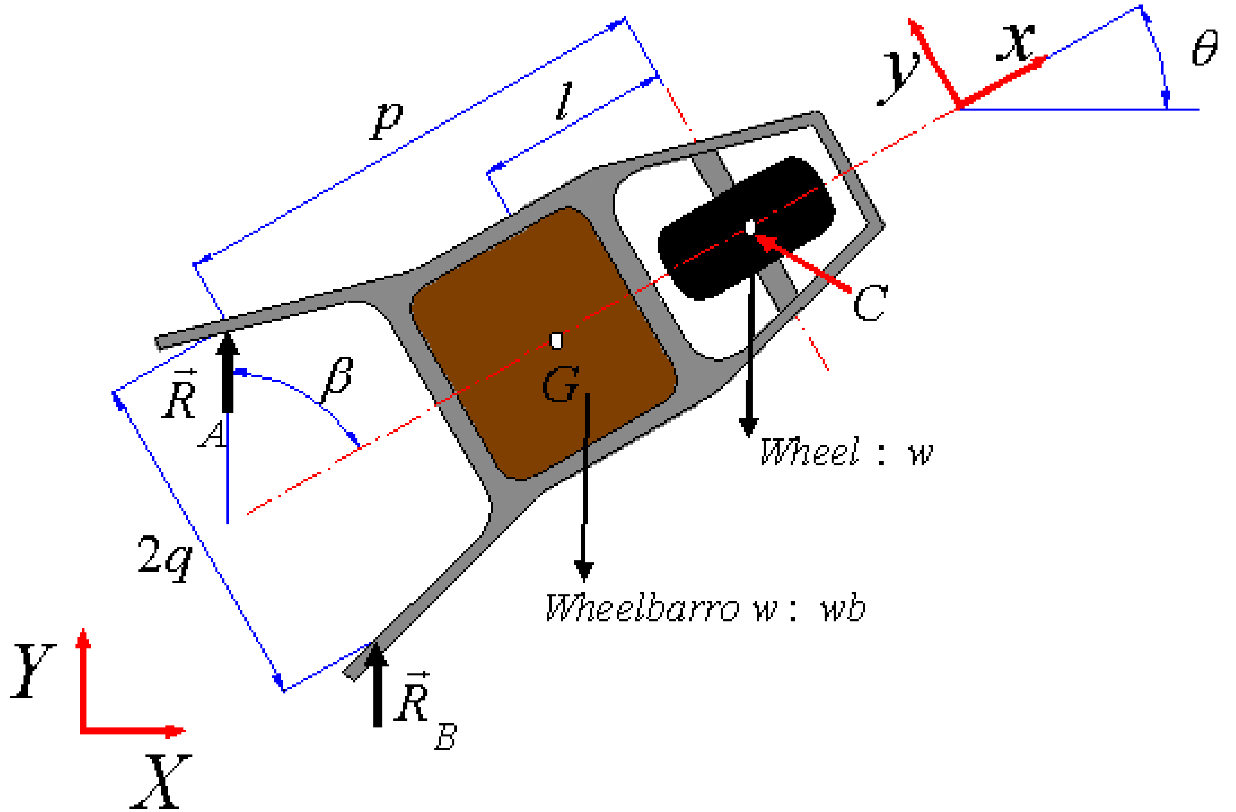
$$(m_1L_1{}^2 + m_2L_3{}^2B(q)^2)\dot{u}_1 + m_2L_3{}^2B(q)\frac{\partial B(q)}{\partial q}Y(q)u_1{}^2 + g(m_1L_1s_1 + m_3L_3s_3B(q)) = \tau$$

The above equation is valuable, because the Lagrange and Newton's methods are not able to achieve this simple form of equations.

In Newton's method, after derivation of equations you must eliminate the internal forces and by using constraints equations you must try to attain the mentioned form of equation.

On the other hand, the Lagrange's method is better than the Newton's method because of elimination of the internal forces in the equations of motion. In Lagrange's method, the number of equations of motion is equal to number of G.C. which is equal or larger than degrees of freedom. So in the presence of constraints, the Lagrange's equations and the constraints relations must be used together to obtain some independent equations.

- **Example 3:** This is a nonholonomic system with 2 degree of freedom and one constraint.



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The selected G.C. for this problem are:

$$q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} \phi \\ \theta \\ X_c \\ Y_c \end{bmatrix}$$

ϕ :The wheel spin angle

θ :The wheelbarrow rotation angle

$X_c \ , \ Y_c$:The wheelbarrow center of mass position

We choose our the generalized speeds as following:

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} v \\ \dot{\theta} \end{bmatrix}$$

According to the problem kinematics we have:

$$\vec{v}^{w} = \vec{v}^{C} = \dot{X}_c \hat{I} + \dot{Y}_c \hat{J} = v\hat{i}$$

$$\dot{X}_c = v\cos(\theta) = u_1 \cos(\theta)$$

$$\dot{Y}_c = v\sin(\theta) = u_1 \sin(\theta)$$

$$\dot{\phi} = \frac{v}{r} = \frac{u_1}{r}$$

$$\vec{\omega}^{wb} = \dot{\theta}\hat{K} = u_2 \hat{K} = u_2 \hat{k}$$

$$\vec{\alpha}^{wb} = \ddot{\theta}\hat{K} = \dot{u}_2 \hat{K}$$

$$\vec{\omega}^{w} = \vec{\omega}^{wb} + \dot{\phi}\hat{j} = \frac{u_1}{r}\hat{j} + u_2 \hat{k}$$

$$\vec{\alpha}^{w} = \frac{\dot{u}_1}{r}\hat{j} + \dot{u}_2 \hat{k} + \vec{\omega}^{wb} \times \vec{\omega}^{w} = -\frac{u_1 u_2}{r}\hat{i} + \frac{\dot{u}_1}{r}\hat{j} + \dot{u}_2 \hat{k}$$

$$\vec{v}^{wb}=\vec{v}^{G}=\vec{v}^{C}+\vec{\omega}^{wb}\times\vec{r}^{G/C}=v\hat{i}-l\dot{\theta}\hat{j}=u_1\hat{i}-lu_2\hat{j}$$

$$\vec{a}^{w}=\vec{a}^{C}=\frac{d(u_1\hat{i})}{dt}=\dot{u}_1\hat{i}+u_1u_2\hat{j}$$

$$\vec{a}^{wb}=\vec{a}^{G}=\vec{a}^{C}+\vec{\alpha}^{wb}\times\vec{r}^{G/C}+\vec{\omega}^{wb}\times(\vec{\omega}^{wb}\times\vec{r}^{G/C})=(\dot{u}_1+lu_2{}^2)\hat{i}+(u_1u_2-l\dot{u}_2)\hat{j}$$

By differentiating, the partial velocities are:

$$\vec{\omega}_1{}^{wb}=0 \qquad \vec{\omega}_2{}^{wb}=\hat{k} \qquad \vec{\omega}_1{}^{w}=\frac{1}{r}\hat{j} \qquad \vec{\omega}_2{}^{w}=\hat{k}$$

$$\vec{v}_1{}^{w}=\hat{i} \qquad \vec{v}_2{}^{w}=0 \qquad \vec{v}_1{}^{wb}=\hat{i} \qquad \vec{v}_2{}^{wb}=-l\hat{j}$$

For determination of the generalized inertia forces, we have:

$$^{xyz}I^{w}=\begin{bmatrix}\frac{J}{2} & 0 & 0\\ 0 & J & 0\\ 0 & 0 & \frac{J}{2}\end{bmatrix}$$

$$\tilde{\vec{M}}^{w}=-(I^{w}\vec{\alpha}^{w}+\vec{\omega}^{w}\times I^{w}\vec{\omega}^{w})$$

$$\tilde{\vec{M}}^{w}=-J(-\frac{u_1u_2}{2r}\hat{i}+\frac{\dot{u}_1}{r}\hat{j}+\frac{\dot{u}_2}{2}\hat{k}-\frac{u_1u_2}{2r}\hat{i})=-J(-\frac{u_1u_2}{r}\hat{i}+\frac{\dot{u}_1}{r}\hat{j}+\frac{\dot{u}_2}{2}\hat{k})$$

$$\vec{\tilde{M}}^{wb} = -(I^{wb}\vec{\alpha}^{wb} + \vec{\omega}^{wb} \times I^{wb}\vec{\omega}^{wb}) = -(I\dot{u}_2\hat{k})$$

$$\vec{\omega}^{w}{}_1\vec{\tilde{M}}^{w} + \vec{\omega}^{wb}{}_1\vec{\tilde{M}}^{wb} = -(J\frac{\dot{u}_1}{r^2}) \qquad \vec{\omega}^{w}{}_2\vec{\tilde{M}}^{w} + \vec{\omega}^{wb}{}_2\vec{\tilde{M}}^{wb} = -(I\dot{u}_2 + J\frac{\dot{u}_2}{2})$$

$$\vec{v}_1{}^{w}.(-m_w\vec{a}^{w}) + \vec{v}_1{}^{wb}.(-m_{wb}\vec{a}^{wb}) = -((m_w + m_{wb})\dot{u}_2)$$

$$\vec{v}_2{}^{w}.(-m_w\vec{a}^{w}) + \vec{v}_2{}^{wb}.(-m_{wb}\vec{a}^{wb}) = -(-lm_{wb}(u_1u_2 - l\dot{u}_2))$$

$$F_1{}^* = -((m_w + m_{wb})\dot{u}_2 + J\frac{\dot{u}_1}{r^2} + m_{wb}lu_2{}^2)$$

$$F_2{}^* = -((I + \frac{J}{2} + m_{wb}l^2)\dot{u}_2 - lm_{wb}u_1u_2)$$

Now the generalized forces must be determined:

$$\vec{v}^{A} = \vec{v}^{C} + \vec{\omega}^{wb} \times \vec{r}^{A/C} = (v - q\dot{\theta})\hat{i} - p\dot{\theta}\hat{j} = (u_1 - qu_2)\hat{i} - pu_2\hat{j}$$

$$\vec{v}^{B} = \vec{v}^{C} + \vec{\omega}^{wb} \times \vec{r}^{B/C} = (v + q\dot{\theta})\hat{i} - p\dot{\theta}\hat{j} = (u_1 + qu_2)\hat{i} - pu_2\hat{j}$$

$$\vec{v}_1{}^{A} = \hat{i} \qquad \vec{v}_2{}^{A} = -q\hat{i} - p\hat{j} \qquad \vec{v}_1{}^{B} = \hat{i} \qquad \vec{v}_2{}^{B} = q\hat{i} - p\hat{j}$$

$$F_1 = \vec{v}_1^{\,A}.\vec{R}^A + \vec{v}_1^{\,B}.\vec{R}^B = 2R\cos(\beta)$$

$$F_2 = \vec{v}_2^{\,A}.\vec{R}^A + \vec{v}_2^{\,B}.\vec{R}^B = -2pR\sin(\beta)$$

$$R^A = R^B = R$$

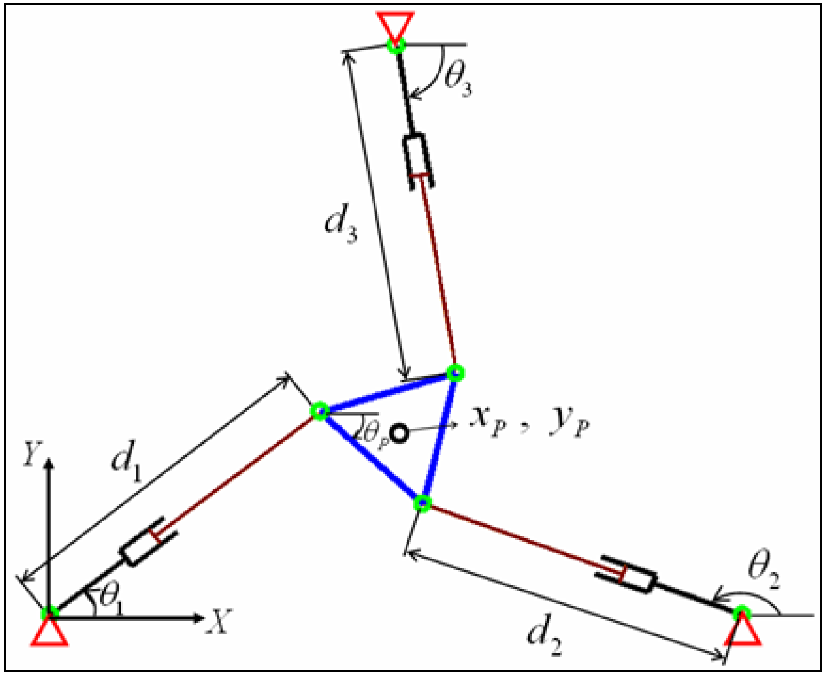
Finally the equations of motion will be:

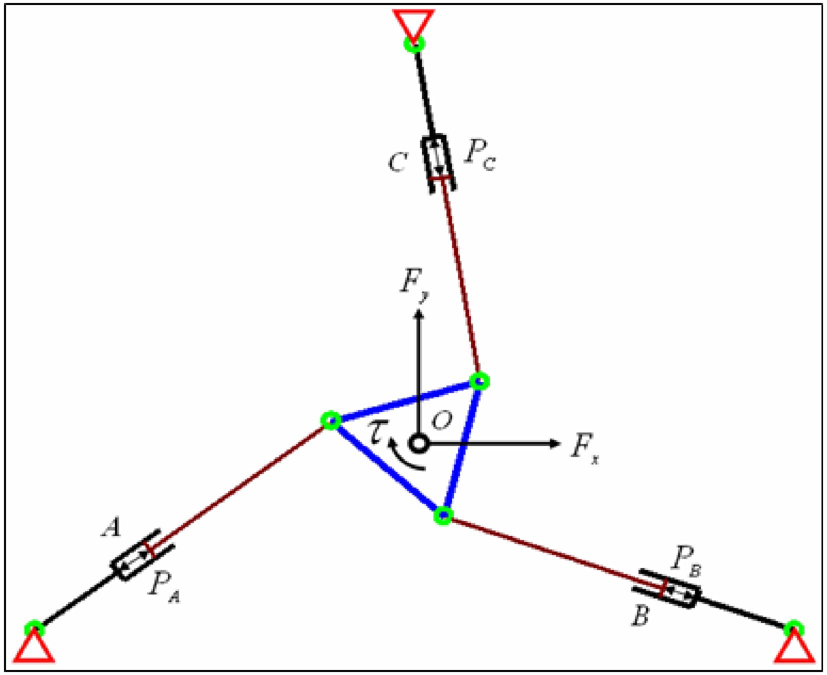
$$r=1: \quad (m_w + m_{wb})\dot{u}_2 + J\frac{\dot{u}_1}{r^2} + m_{wb} l u_2^{\,2} = 2R\cos(\beta)$$

$$r=2: \quad (I + \frac{J}{2} + m_{wb} l^2)\dot{u}_2 - l m_{wb} u_1 u_2 = -2pR\sin(\beta)$$

The simple equations determined by Kane's method can not be obtained from Lagrange's method. Although the Lagrange's method reach the equations of motion simpler than the Kane's method, the final form of equations in Kane's method is very

➢ **Example 4:** This example is a 2D Stewart mechanism with three degrees of freedom called planner 3RPR. The positive angular direction is counter clockwise. This is a holonomic system with complicated constraints.





The introductory step in any problem is to define suitable generalized coordinates in order to decrease the complexity of the problem. These definitions are presented as:

$$q_1 = \theta_1, \quad q_2 = \theta_2, \quad q_3 = \theta_3$$
$$q_4 = \theta_p, \quad q_5 = x_p, \quad q_6 = y_p$$
$$q_7 = d_1, \quad q_8 = d_2, \quad q_9 = d_3$$

Since the mechanism has 3DOF and we have defined 9 GC here, 6 holonomic constraint equations should confine the motion. We choose the generalized speeds as:

$$u_1 = \dot{x}_p, \quad u_2 = \dot{y}_p, \quad u_3 = \dot{\theta}_p$$

For each link of the 3RPR, an open loop chain between the basis and point P is written. For instance, the first open chain is:

$$\dot{q}_7 \cos q_1 - q_7 \dot{q}_1 \sin q_1 = \dot{q}_5 + \rho \dot{q}_4 \sin(\beta_1 + q_4)$$
$$\dot{q}_7 \sin q_1 + q_7 \dot{q}_1 \cos q_1 = \dot{q}_6 - \rho \dot{q}_4 \cos(\beta_1 + q_4)$$

We could also eliminate the position of center of triangle (x,y) and reduce the number of GC to 7. Superficially, it might seem to be the better solution but the complexity of the problem would be more than before and the CPU time would increase up to 10 times more.

We have to evaluate vector $\{v_k\}$ and $\{\omega_k\}$ for all of 7 bodies of the system. for example for the first link:

$$\{V_1\} = \frac{L_1}{2}\dot{q}_1 \begin{Bmatrix} -\sin q_1 \\ \cos q_1 \end{Bmatrix} \qquad \omega_1 = \dot{q}_1$$

By substituting $\dot{q}_1$ we have:

$$\{V_1\} = \frac{L_1}{2q_7}(-(u_1 + \rho u_3 \sin(\beta_1 + q_4))\sin q_1 + (u_2 - \rho u_3 \cos(\beta_1 + q_4))\cos q_1) \begin{Bmatrix} -\sin q_1 \\ \cos q_1 \end{Bmatrix}$$

$$\omega_1 = \frac{-(u_1 + \rho u_3 \sin(\beta_1 + q_4))\sin q_1}{q_7} + \frac{(u_2 - \rho u_3 \cos(\beta_1 + q_4))\cos q_1}{q_7}$$

And the partial derivative with respect to the generalized speeds and the generalized coordinates will be:

$$\frac{\partial\{v_1\}}{\partial\{u\}}(:,1)=\frac{L_1S_1}{2q_7}\begin{bmatrix}S_1\\-C_1\end{bmatrix}$$

$$\frac{\partial\{v_1\}}{\partial\{u\}}(:,2)=\frac{L_1C_1}{2q_7}\begin{bmatrix}-S_1\\C_1\end{bmatrix}$$

$$\frac{\partial\{v_1\}}{\partial\{u\}}(:,3)=\frac{\rho L_1C_2}{2q_7}\begin{bmatrix}S_1\\-C_1\end{bmatrix}$$

$$\frac{\partial\omega_1}{\partial\{u\}}=\frac{1}{q_7}\begin{bmatrix}-S_1 & C_1 & -\rho C_2\end{bmatrix}$$

$$\frac{\partial\{v_1\}}{\partial\{q\}}(:,1)=\frac{L_1}{2q_7}\begin{bmatrix}u_1S_3+\rho u_3C_4-u_2C_3\\-u_1C_3+\rho u_3C_4-u_2S_3\end{bmatrix}$$

$$\frac{\partial\{v_1\}}{\partial\{q\}}(:,4)=\frac{\rho u_3L_1S_2}{2q_7}\begin{bmatrix}S_1\\-C_1\end{bmatrix}$$

$$\frac{\partial\{v_1\}}{\partial\{q\}}(:,7)=\frac{L_1(-u_1S_1-\rho u_3C_2+u_2C_1)}{2q_7^2}\begin{bmatrix}S_1\\-C_1\end{bmatrix}$$

$$\frac{\partial\{v_1\}}{\partial\{q\}}(:,j)=\begin{bmatrix}0\\0\end{bmatrix}\qquad j\neq 1,4,7$$

$$\frac{\partial\omega_1}{\partial\{q\}}(:,1)=\frac{-1}{q_7}(u_1C_1+\rho u_3S_2+u_2S_1)$$

$$\frac{\partial\omega_1}{\partial\{q\}}(:,4)=\frac{-1}{q_7}\rho u_3S_2$$

$$\frac{\partial\omega_1}{\partial\{q\}}(:,7)=\frac{u_1S_1+\rho u_3C_2-u_2C_1\cos q_1}{q_7^2}$$

$$\frac{\partial\omega_1}{\partial\{q\}}(:,j)=\begin{bmatrix}0\\0\end{bmatrix}\qquad j\neq 1,4,7$$

where we have:

$$C_1 = \cos(q_1), \qquad S_1 = \sin(q_1)$$
$$C_2 = \cos(q_4 + \beta_1 + q_1), \qquad S_2 = \sin(q_4 + \beta_1 + q_1)$$
$$C_3 = \cos(2q_1), \qquad S_3 = \sin(2q_1)$$
$$C_4 = \cos(q_4 + \beta_1 + 2q_1), \qquad S_4 = \sin(q_4 + \beta_1 + 2q_1)$$

To obtain the generalized force we have:

$$F(j) = \{P_A\}^T \frac{\partial\{V_A\}}{\partial u_j} + \{P_B\}^T \frac{\partial\{V_B\}}{\partial u_j} + \{P_C\}^T \frac{\partial\{V_C\}}{\partial u_j} + \{F_x \quad F_y\} \frac{\partial\{V_O\}}{\partial u_j} - \tau \frac{\partial \omega_P}{\partial u_j}$$

The Kane's method efficacy is proved if one implements this method by using the computer programming.

متشکرم

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