بسم الله الرحمن الرحیم

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Lagrangian Equation of Motion

Purpose:

- **To extend the *Energy* approach in deriving equations of motion (i.e. Lagrange's Method) for Mechanical Systems.**

Topics:

- **Generalized Coordinates**
- **Lagrangian Equation of Motion for *Independent* Set of Generalized Coordinates**
- **Lagrangian Equation of Motion for *Dependent* Set of Generalized Coordinates**
- **Hamiltonian Principle**

Lagrangian Equation of Motion
for
(*Dependent Set of Generalized Coordinates*)

When functional relations exist among the generalized coordinates, the Lagrange's Equation must include the Constraint Relations.

***Definition*: A Constrained Generalized Coordinate Set has the generalized coordinates related by a system of functions, $g_r(q^m)=0$, called Constraints.**

$$g_r(q^1, \ldots, q^N) = 0, \quad r = 1, \ldots, R \qquad \textbf{(11.16)}$$

***Degrees-of-Freedom* =**

# of Dependent Generalized Coordinates - # of Constraints

Constraints are occasionally represented in differential forms (i.e. most common is the functional relationship among generalized velocities).

***Concept of Constraint*:**

(a) Integral Expression (Integrated form of constraint; *Holonomic* Constraint):

$$g_r(q^m, t) = 0, \quad r = 1, \ldots, R; \quad m = 1, \ldots, N \qquad \textbf{(11.17)}$$

(b) Differential Expression (Differentiated form of constraint; *Non-Holonomic* Constraint)

$$\dot{g}_r = \frac{\partial g_r}{\partial q^m}\dot{q}^m + \frac{\partial g_r}{\partial t} = 0, \quad r = 1, \ldots, R < N; \quad m = 1, \ldots, N$$

$$C_{rm}\dot{q}^m + C_r = 0 \quad \Rightarrow \quad C_{rm}dq^m + C_r dt = 0 \qquad \textbf{(11.18)}$$

$$\{C_{rm}\} = \frac{\partial g_r}{\partial q^m}$$ **(*Constraint Coefficients*; not constant and function of q^m)**

In Virtual Form:

$$\delta g_r = \frac{\partial g_r}{\partial q^m}\delta q^m + \frac{\partial g_r}{\partial t}\cancel{\delta t} = 0$$

(*when δt is set to be zero by virtual concept*)

$$\delta g_r = C_{rm}\delta q^m = 0 \qquad \textbf{(11.19)}$$

Equation (11.19) is an *Integrated* or *Differential* form of constraint expressed by virtual displacement.

If it is possible to go from (11.19) to (11.17), then the system is *Holonomic*. (Greek word meaning Integrable)

If it is not possible to go from (11.19) to (11.17), then the system is *Non-Holonomic*.

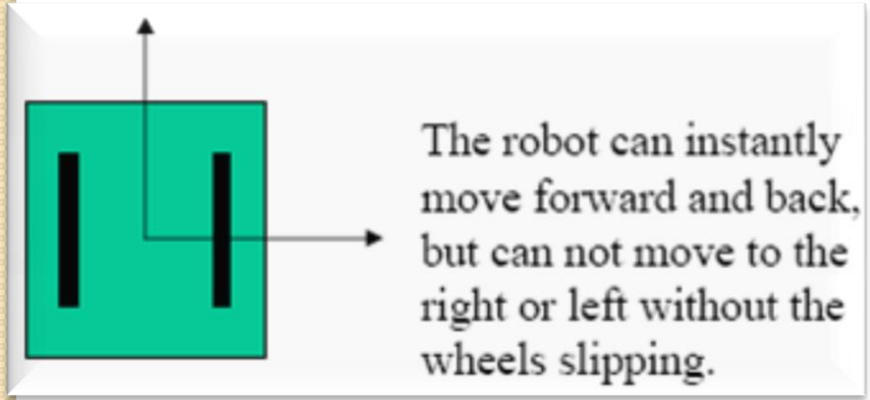
Definition **(Holonomic System): When the constraints of a system are in the integrated (integrable differentia) form, the number of Generalized Coordinates can be reduced to correspond to the Degrees-of-Freedom of the system.**

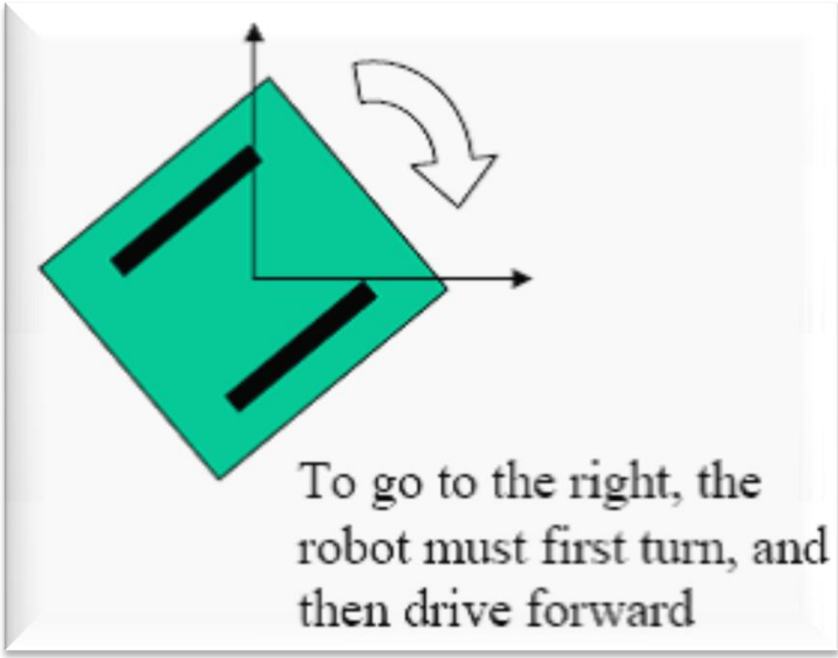
Definition **(Non-Holonomic System): When some of the constraints of the system are in the non-integrable (differential) form, the number of Generalized Coordinates can not be reduced to correspond to the Degrees-of-Freedom of the system.**

Equation (11.19): $C_{rm}\delta q^{m} = 0$ **implies that {δq^m} is no longer an independent set.**

Example: Non-Holonomic Constraints often arise in systems having parts that roll.

Example: (Non-Holonomic Constraint) ***A constraint on velocity does not induce a constraint on position.*** **For a wheeled robot, it can instantaneously move in some directions (forwards and backwards), but not others (side to side).**



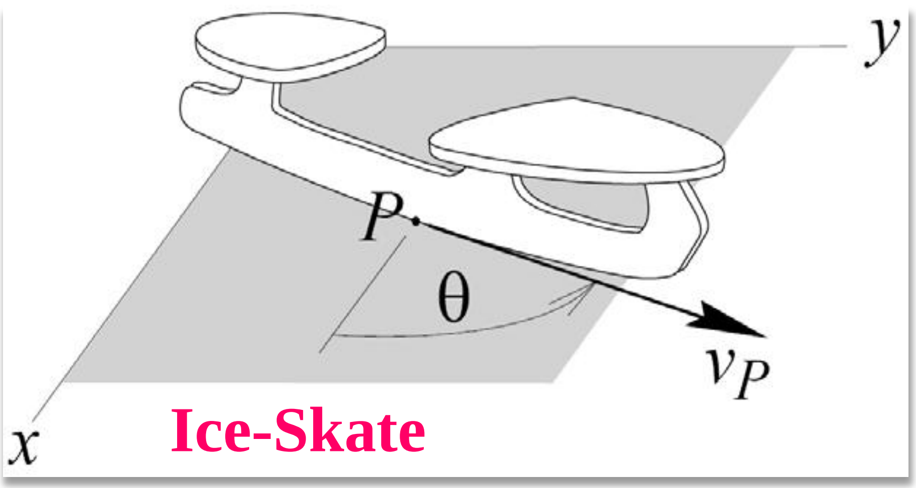


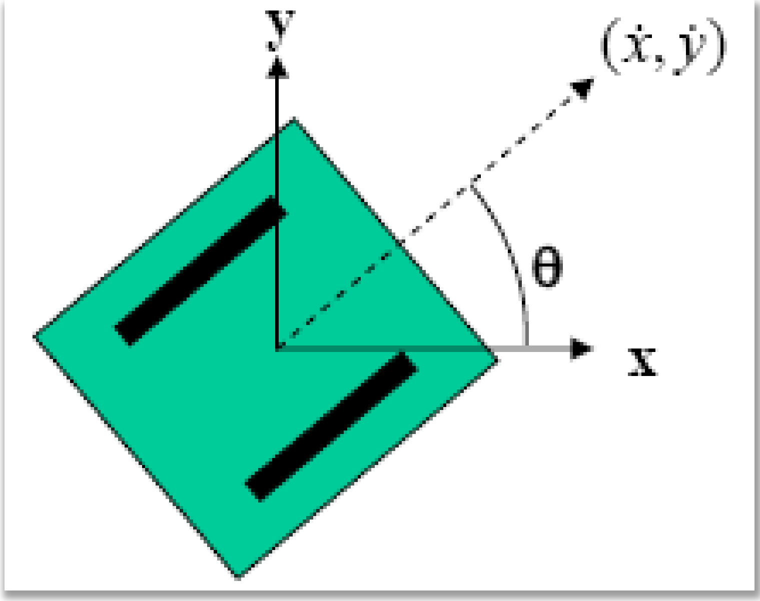
Mathematical Description of the Constraint: For a differential drive, this is:

$$\boxed{\dot{y}\cos\theta - \dot{x}\sin\theta = 0}$$

The directions car can't move are:

1. **If $\theta = 0$, then velocity in $y = 0$, and**
2. **If $\theta = 90$, then velocity in $x = 0$.**





Example: (Holonomic System)

A Person Walking**; you can instantly step to the left and right, as well as going forward and backward. In other words, your velocity in plane is not restricted.**

An Omni wheeled robot; is another example of Holonomic System. It can roll forwards and backwards, as well as sideways.

Recall: since δq^m is no longer an independent set, then its coefficient in equation (11.15) can not be set to zero. Therefore, the Lagrange's Equation in the above form can not be applied to *Constraint (Dependent) form of Generalized Coordinates*.

$$\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^m}\right)-\frac{\partial T}{\partial q^m}-Q_m\right]\delta q^m=0 \qquad \textbf{(11.15)}$$

Theorem-44: The dynamic process of a mechanical system, described in a set of constrained generalized coordinates, must satisfy the following set of equations:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^m}\right)-\frac{\partial T}{\partial q^m}+C_{rm}\lambda^r=Q_m \qquad \begin{cases} m=1,\dots,N \\ r=1,\dots,R \end{cases} \qquad \textbf{(11.20)}$$

Where: λ^r are R-number of Lagrangian Multipliers corresponding to the number of constraints.

Proof: **Lets add the null combination of *Lagrangian Multipliers* and the constraints to Equation (11.15) as:**

$$\lambda^r C_{rm} \delta q^m = 0$$

$$[\frac{d}{dt}(\frac{\partial T}{\partial \dot{q}^m}) - \frac{\partial T}{\partial q^m} + \lambda^r C_{rm} - Q_m]\delta q^m = 0 \qquad \begin{cases} m = 1,\dots,N \\ r = 1,\dots,R \end{cases} \tag{11.21}$$

Let us now separate Equation (11.21) into two groups:

1. **The linear combination of those dependent variables $\{\delta q^1,\dots,\delta q^R\}$, and**
2. **The linear combination of those independent set $\{\delta q^{R+1},\dots,\delta q^N\}$.**

$$[\frac{d}{dt}(\frac{\partial T}{\partial \dot{q}^s})-\frac{\partial T}{\partial q^s}+\lambda^r C_{rs}-Q_s]\delta q^s +$$

$$+[\frac{d}{dt}(\frac{\partial T}{\partial \dot{q}^k})-\frac{\partial T}{\partial q^k}+\lambda^r C_{rk}-Q_k]\delta q^k = 0 \qquad ;\begin{cases} s=1,\dots,R \\ k=R+1,\dots,N \end{cases}$$

(11.22)

Now, we can impose to choose "R" number of "λ's" so that R equations become zero. In other words, a set of $\{\lambda^r\}$ can be found such that the coefficients of $\{\delta q^s\}$ in equation (11.22) become zero.

$$\frac{d}{dt}(\frac{\partial T}{\partial \dot{q}^s})-\frac{\partial T}{\partial q^s}+\lambda^r C_{rs}=Q_s \qquad (11.23)$$

(For a special set of $\{\lambda^r\}$ with dependent variables)

What remains from Equation (11.22), are the second group, as:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^{k}}\right)-\frac{\partial T}{\partial q^{k}}+\lambda^{r}C_{rk}=Q_{k} \qquad \textbf{(11.24)}$$

(For an independent $\{\delta q^{k}\}$)

Equations (11.23) and (11.24) together form a complete set for Equation (11.20).

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^{m}}\right)-\frac{\partial T}{\partial q^{m}}+C_{rm}\lambda^{r}=Q_{m} \qquad \begin{cases} m=1,\dots,N \\ r=1,\dots,R \end{cases}$$

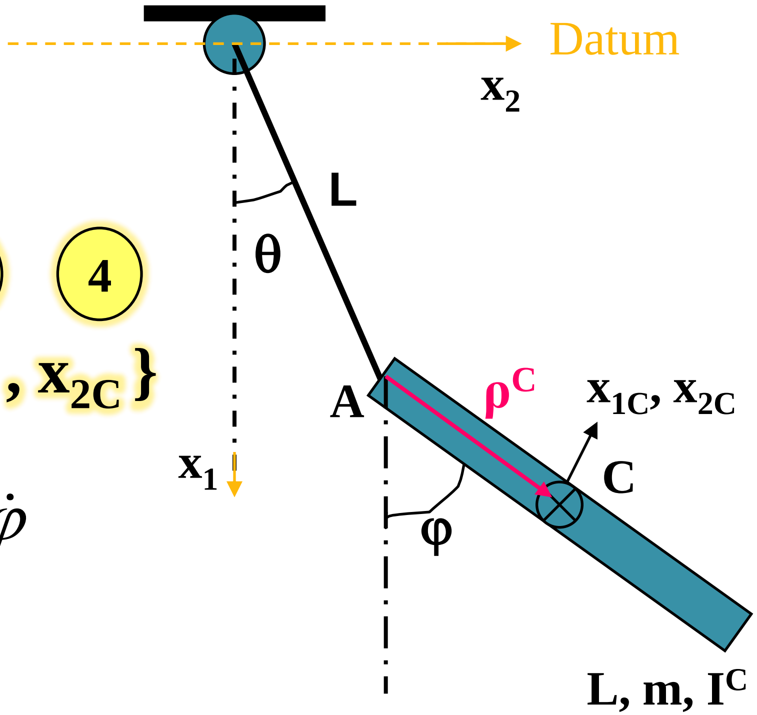
(11.20)

Conceptually, the *Lagrangian Multipliers* help to reduce the dependent N-set of generalized coordinates into the Independent (N-R) set.

In other words, the introduction of the Lagrangian Multipliers expands the space to (N+R) for $\{q^m, \lambda^r\}$ without constraints.

(N+R) = set of equations

Example: **Express Differential Equations of Motion of the following system:**



1. Motion:

1 2 3 4

Let's choose $\{q^m\} = \{\theta, \varphi, x_{1C}, x_{2C}\}$

$$(v^C)^2 = (\dot{x}_{1C})^2 + (\dot{x}_{2C})^2, \quad \omega = \dot{\varphi}$$

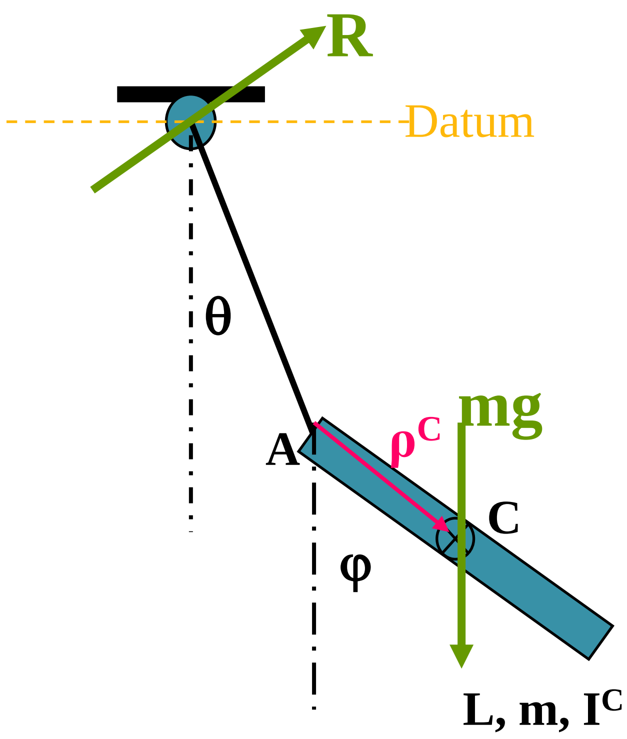
2. Kinetic Energy:

$$T = \frac{1}{2} m[(\dot{x}_{1C})^2 + (\dot{x}_{2C})^2] + \frac{1}{2} I^C \dot{\varphi}^2$$

3. Virtual Work: Free Body Diagram

$$\delta U = mg\,\delta x_{1C}$$

$$\{Q_m\} = \begin{Bmatrix} 0 \\ 0 \\ mg \\ 0 \end{Bmatrix}$$



4. Constraints:

$$g_1 = x_{1C} - L\cos\theta - \rho^C \cos\varphi = 0 \qquad \textbf{(1)}$$

$$g_2 = x_{2C} - L\sin\theta - \rho^C \sin\varphi = 0 \qquad \textbf{(2)}$$

$$g_1 = x_{1C} - L\cos\theta - \rho^C \cos\varphi = 0 \qquad \textbf{(1)}$$

$$g_2 = x_{2C} - L\sin\theta - \rho^C \sin\varphi = 0 \qquad \textbf{(2)}$$

$$C_{rm} = \frac{\partial g_r}{\partial q^m}$$

C_{rm}

r \ m	θ	φ	x_{1C}	x_{2C}
1	$L\sin\theta$	$\rho^C\sin\varphi$	1	0
2	$-L\cos\theta$	$-\rho^C\cos\varphi$	0	1

5. Lagrangian Equation of Motion:

5.1 For Coordinate "q¹ = θ";

$$\frac{d}{dt}\left(\cancelto{0}{\frac{\partial T}{\partial \dot{\theta}}}\right)-\cancelto{0}{\frac{\partial T}{\partial \theta}}+\lambda^{1}C_{1\theta}+\lambda^{2}C_{2\theta}=Q_{\theta}$$

$$\lambda^{1}(L\sin\theta)+\lambda^{2}(-L\cos\theta)=0 \qquad \textbf{(3)}$$

5.2 For Coordinate "q² = φ";

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\varphi}}\right)-\frac{\partial T}{\partial \varphi}+\lambda^{1}C_{1\varphi}+\lambda^{2}C_{2\varphi}=Q_{\varphi}$$

$$I^{C}\ddot{\varphi}+\lambda^{1}(\rho^{C}\sin\varphi)+\lambda^{2}(-\rho^{C}\cos\varphi)=0 \qquad \textbf{(4)}$$

5.3 For Coordinate "$q^3 = x_{1C}$";

$$m\ddot{x}_{1C} + \lambda^1(1) = mg \qquad (5)$$

5.4 For Coordinate "$q^4 = x_{2C}$";

$$m\ddot{x}_{2C} + \lambda^2(1) = 0 \qquad (6)$$

We are faced with six equations and six unknowns, that can be solved for {θ, φ, x_{1C}, x_{2C}, λ^1, λ^2}.

Constraint equations and their derivatives are used and substituted in equations (5) and (6) till x_{1C}, x_{2C}, λ^1, λ^2 are eliminated from all equations, and two independent equations in terms θ and φ and their derivatives are remained.

متشکرم