بسم الله الرحمن الرحیم

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Lagrangian Equation of Motion

***Purpose*:**

- **To extend the *Energy* approach in deriving equations of motion (i.e. Lagrange's Method) for Mechanical Systems.**

***Topics*:**

- **Generalized Coordinates**
- **Lagrangian Equation of Motion for *Independent* Set of Generalized Coordinates**
- **Lagrangian Equation of Motion for *Dependent* Set of Generalized Coordinates**
- **Hamiltonian Principle**

Generalized Coordinates

Definition: A "*Generalized Coordinate System*" is any set of variables $\{q^m\}$ which defines the position of a mechanical system. Generalized coordinates are usually chosen as the best description of the mechanical configuration (i.e. Polar, Cylindrical, Spherical, etc.).

Definition: "*Degrees-of-Freedom*" is the minimum number of generalized coordinates which adequately define the position of the mechanical system.

Example:

- A free **Particle** in space has: 3-DOF
- A free **Rigid Body** in space has: 6-DOF
- A **Structure** has: 0-DOF

An analyst for convenience may choose to use more generalized coordinates than the number of degrees-of-freedom of the system. In this case, the set of generalized coordinates is not independent. Therefore:

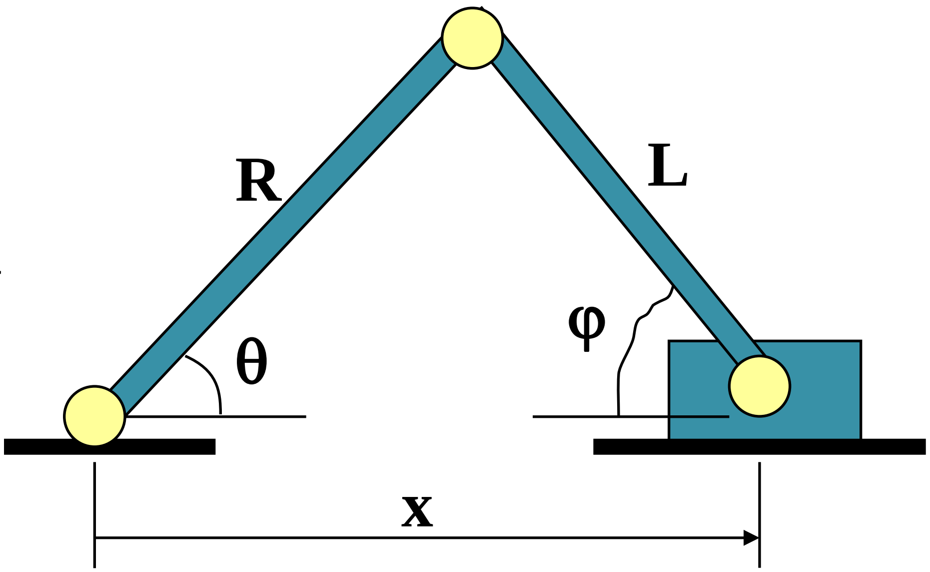
An Independent Coordinate Set: exists when the number of generalized coordinates correspond to the degree-of-freedom, and

A Dependent Coordinate Set (Constraints): exists when we have a number of functional relations among the generalized coordinates.

Example: **Consider the one degree-of-freedom system shown.**

$\{\theta\}$: **Independent Set**

$\{\theta, \varphi, x\}$: **Dependent Set**



***Constraints* are:**

$$\begin{cases} g_1 = R\sin\theta - L\sin\varphi = 0 \\ g_2 = R\cos\theta + L\cos\varphi - X = 0 \end{cases}$$

$$g_1 \quad and \quad g_2 \ functions \quad of : \{q^m\}$$

Definition: $\underline{\dot{q}}$ or $\{\dot{q}^m\}$ **is defined as the** ***Generalized Velocity*****.**

Definition: $\delta \underline{q}$ or $\{\delta q^m\}$ **is defined as the** ***Generalized Virtual Displacement*****.**

Definition: ***The Generalized Momentum*** **$\{P_m\}$ is the**

Set $\{\frac{\partial T}{\partial \dot{q}^m}\}$

$$\{P_m\} = \{\frac{\partial T}{\partial \dot{q}^m}\} \qquad \textbf{(11.1)}$$

Note that: In the Newtonian Mechanics, the momentum of βth particle is:

$$P_i^\beta = \frac{\partial T}{\partial \dot{x}_i^\beta} \qquad \textbf{(11.2)}$$

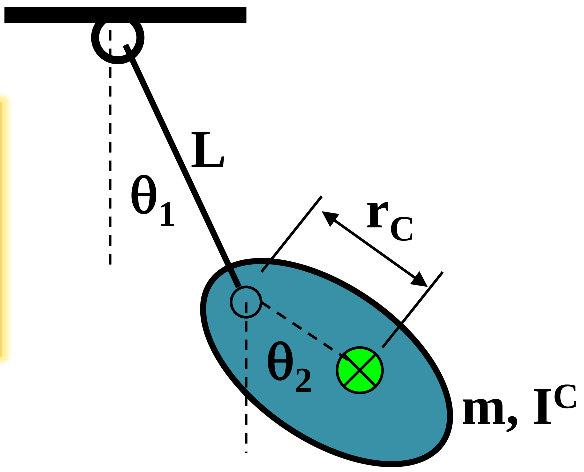
Definition: The ***Generalized Force*** **Q** ={Q_m} is the set of coefficients of generalized virtual displacement {δq^m} in the expression of virtual work as:

$$\delta U = Q_m \delta q^m \qquad (11.3)$$

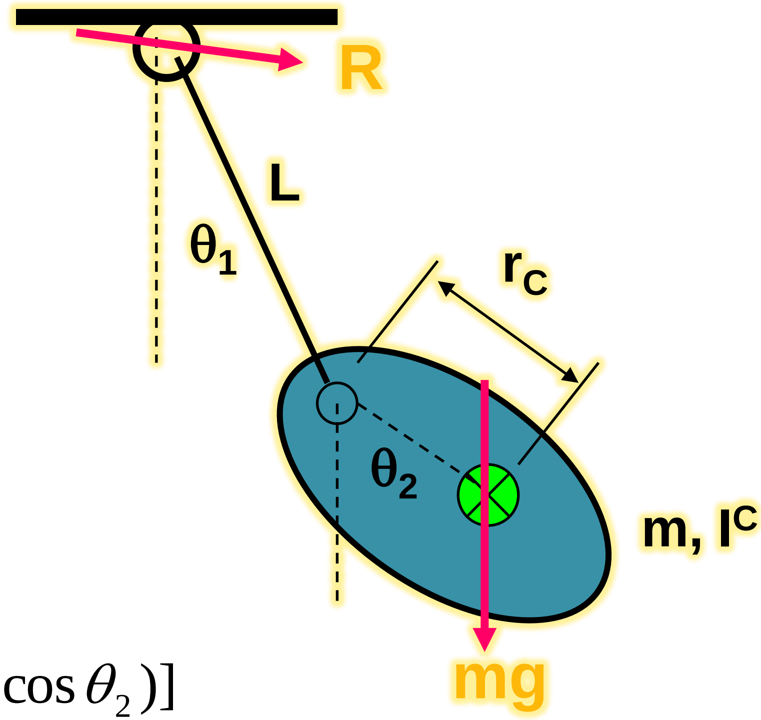
Note that: For a system of N particles, the working force in the expression of work is:

$$\delta U = \sum_{\beta=1}^{N} f_i^{\beta} \delta x_i^{\beta} \qquad (11.4)$$

Example: Determine the Generalized Forces for the double pendulum shown?



Work done from equilibrium position to the existing position is:



$$U = -mg[L(1-\cos\theta_1) + r_C(1-\cos\theta_2)]$$

$$\delta U = -mgL\sin\theta_1\delta\theta_1 - mgr_C\sin\theta_2\delta\theta_2$$

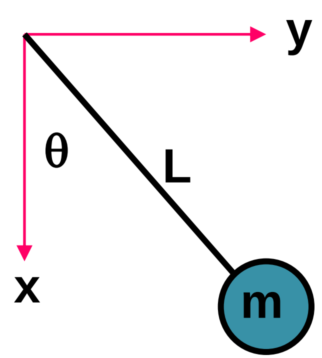
Hence:

$$Generalized \quad Force \equiv \underline{Q} = \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \begin{Bmatrix} -mgL\sin\theta_1 \\ -mgr_C\sin\theta_2 \end{Bmatrix}$$

Some Useful Relations: **Functional relations between Cartesian coordinates and generalized coordinates;**

1). $$\begin{cases} \{x_i\} \overset{f_i}{\longleftrightarrow} \{q^m\} \\ x_i = f_i(q^m) \end{cases} \qquad \textbf{(11.5)}$$

Ex: $\begin{cases} x = L\cos\theta \\ y = L\sin\theta \end{cases}$



$$q^m : Generalized - Coordinates$$

2). By Chain Rule expansion of derivatives:

$$\dot{x}_i = \frac{\partial x_i}{\partial q^m}\dot{q}^m = \frac{\partial x_i}{\partial q^1}\dot{q}^1 + \frac{\partial x_i}{\partial q^2}\dot{q}^2 + \ldots\ldots \qquad \textbf{(11.6)}$$

3). : $$\delta x_i = \frac{\partial x_i}{\partial q^m} \delta q^m \qquad (11.7)$$

4). From relation (11.6), since $\dot{x}_i$ is a linear function of $\dot{q}^m$, we have:

$$\frac{\partial \dot{x}_i}{\partial \dot{q}^m} = \frac{\partial x_i}{\partial q^m} \qquad (11.8)$$

5). By Chain Rule expansion of derivatives, we can write:

$$\frac{d}{dt}\left(\frac{\partial x_i}{\partial q^m}\right) = \frac{\partial \dot{x}_i}{\partial q^m} \qquad (11.9)$$

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Proof:

$$\frac{d}{dt}\left(\frac{\partial x_i}{\partial q^m}\right) = \frac{\partial}{\partial q^r}\left(\frac{\partial x_i}{\partial q^m}\right)\dot{q}^r = \frac{\partial^2 x_i}{\partial q^m \partial q^r}\dot{q}^r =$$

But q^m **is independent of** $\dot{q}^r$

$$= \frac{\partial}{\partial q^m}\Big(\underbrace{\frac{\partial x_i}{\partial q^r}\dot{q}^r}_{from \quad Eq.(11.6) = \dot{x}_i}\Big) = \frac{\partial \dot{x}_i}{\partial q^m}$$

Lagrangian Equation of Motion

for

(*Independent Set of Generalized Coordinates*)

A mechanical system classified according to the coordinate description generally fall under one of the following categories:

***Rheonomic Systems*: The description of configuration by the generalized coordinates varies with time throughout the dynamic process.**

$$x_i = f_i(q^m, t) \qquad q^m = Gen. \quad Coords. \tag{11.10}$$

***Scleronomic Systems*: The description is not explicitly time dependent.**

$$x_i = f_i(q^m) \tag{11.11}$$

***Theorem-43*:** **Throughout the dynamic process of a mechanical system, the motion in the Independent Set of generalized coordinates $\{q^m\}$ will satisfy the equation:**

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^m}\right)-\frac{\partial T}{\partial q^m}=Q_m \qquad m=1,2,\ldots,N \qquad \textbf{(11.12)}$$

Proof: **Based on *Leibniz Principle*;** **$\delta T - \delta U = 0$**

In $\{x_i\}$ space: **Kinetic Energy expression for a single particle is;**

$$T=\frac{1}{2}m\dot{x}_i\dot{x}_i \Rightarrow$$

$$\dot{T}=\frac{d}{dt}\left(\frac{1}{2}m\dot{x}_i\dot{x}_i\right)=m\ddot{x}_i\dot{x}_i=\frac{d}{dt}(m\dot{x}_i)\dot{x}_i=\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_i}\right)\dot{x}_i$$

$$\dot{T} = \frac{d}{dt}\left(\frac{1}{2} m\dot{x}_i \dot{x}_i\right) = m\ddot{x}_i \dot{x}_i = \frac{d}{dt}(m\dot{x}_i)\dot{x}_i = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_i}\right)\dot{x}_i$$

$$dT = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_i}\right)dx_i$$

$$\delta T = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_i}\right)\delta x_i$$

{Virtual Change in K.E. in terms of Virtual Displacement} **(11.13)**

Going to {q^m} space: **Since by eq. (11.7);** $\delta x_i = \frac{\partial x_i}{\partial q^m}\delta q^m$

$$\delta T = \left[\underbrace{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_i}\right)}_{U}\underbrace{\frac{\partial x_i}{\partial q^m}}_{V}\right]\delta q^m$$

$$but : (dU)V = d(UV) - UdV$$

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$$\delta T=\Big[\underbrace{\frac{d}{dt}\Big(\frac{\partial T}{\partial \dot{x}_i}\Big)}_{U}\underbrace{\frac{\partial x_i}{\partial q^m}}_{V}\Big]\delta q^m\,, \qquad but:(dU)V=d(UV)-UdV$$

$$\delta T=\Big[\frac{d}{dt}\Big(\frac{\partial T}{\partial \dot{x}_i}\underbrace{\frac{\partial x_i}{\partial q^m}}_{\text{Eq.(11.8)}\ =\frac{\partial \dot{x}_i}{\partial \dot{q}^m}}\Big)-\frac{\partial T}{\partial \dot{x}_i}\underbrace{\frac{d}{dt}\Big(\frac{\partial x_i}{\partial q^m}\Big)}_{\text{Eq.(11.9)}\ =\frac{\partial \dot{x}_i}{\partial q^m}}\Big]\delta q^m$$

$$\delta T=\Big[\frac{d}{dt}\Big(\frac{\partial T}{\cancel{\partial \dot{x}_i}}\frac{\cancel{\partial \dot{x}_i}}{\partial \dot{q}^m}\Big)-\frac{\partial T}{\cancel{\partial \dot{x}_i}}\frac{\cancel{\partial \dot{x}_i}}{\partial q^m}\Big]\delta q^m$$

$$\delta T=\Big[\frac{d}{dt}\Big(\frac{\partial T}{\partial \dot{q}^m}\Big)-\frac{\partial T}{\partial q^m}\Big]\delta q^m \qquad \textbf{(11.14)}$$

$$\delta T = \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^m}\right) - \frac{\partial T}{\partial q^m}\right]\delta q^m \qquad \textbf{(11.14)}$$

$$But: \begin{cases} \delta U = \delta T & (From \quad Leibniz \quad Principle) \\ \delta U = Q_m \delta q^m & (Def. \quad of \; Generalized \quad Force) \end{cases}$$

(11.3)

From Equations (11.14) and (11.3) we have:

$$\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^m}\right) - \frac{\partial T}{\partial q^m} - Q_m\right]\delta q^m = 0 \qquad \textbf{(11.15)}$$

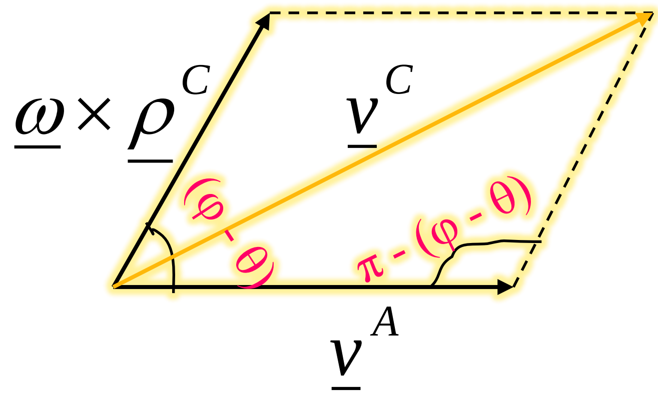
Since δq^m is independent and arbitrary, then:

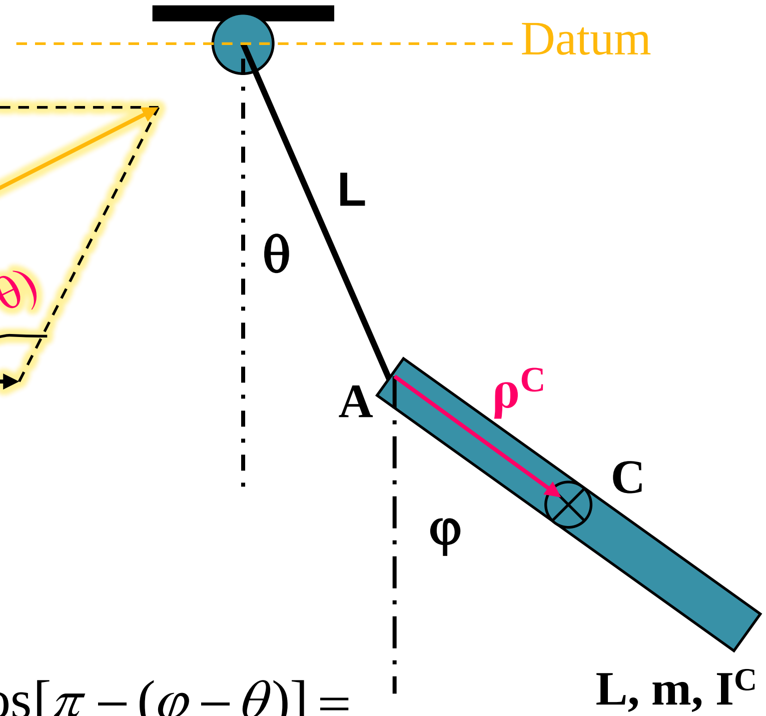
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^m}\right) - \frac{\partial T}{\partial q^m} = Q_m \qquad m = 1,2,\ldots N \qquad \textbf{(11.12)}$$

Lagrangian Equation of Motion for Independent Set of Generalized Coordinates. Applies to both Scleronomic and Rheonomic Systems.

Example: Express Differential Equations of Motion of the following system:

1. Motion:





$$\underline{v}^C=\underline{v}^A+\underline{\omega}\times\underline{\rho}^C\,,\quad \omega=\dot{\varphi}$$

$$(v^C)^2=(\ell\dot{\theta})^2+(\frac{\ell}{2}\dot{\varphi})^2-2(\ell\dot{\theta})(\frac{\ell}{2}\dot{\varphi})\cos[\pi-(\varphi-\theta)]=$$

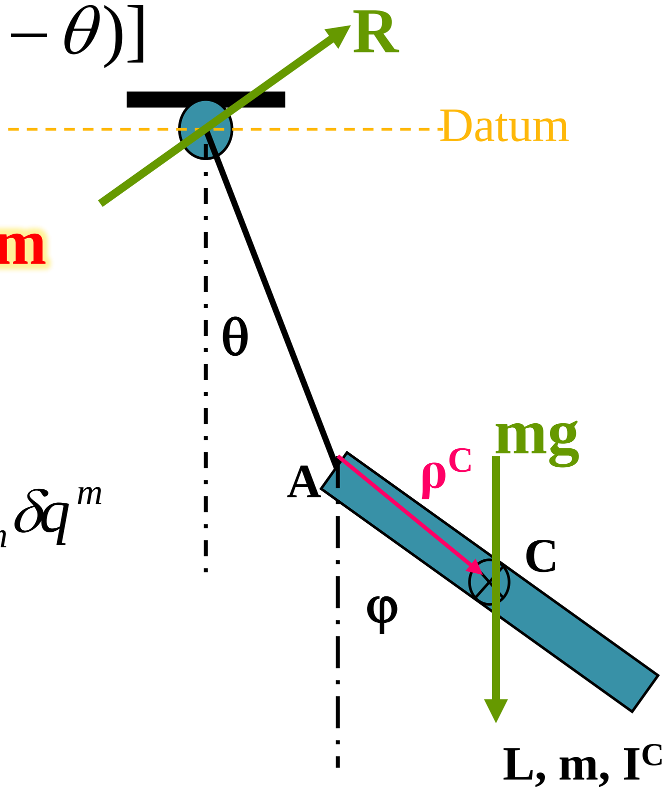
$$=(\ell^2\dot{\theta}^2)+\frac{1}{4}\ell^2\dot{\varphi}^2+\ell^2\dot{\theta}\dot{\varphi}\cos(\varphi-\theta)$$

2. Kinetic Energy:

$$T = T^{(e)} + T^{(i)} = \frac{1}{2} m v_C^2 + \frac{1}{2} I^C \omega^2$$

$$T = \frac{1}{2} m[\ell^2 \dot{\theta}^2 + \frac{1}{4}\ell^2 \dot{\varphi}^2 + \ell^2 \dot{\theta}\dot{\varphi}\cos(\varphi - \theta)] + \frac{1}{2}(\frac{m\ell^2}{12})\dot{\varphi}^2$$

$$T = \frac{1}{2} m[\ell^2 \dot{\theta}^2 + \frac{1}{3}\ell^2 \dot{\varphi}^2 + \ell^2 \dot{\theta}\dot{\varphi}\cos(\varphi - \theta)]$$



3. Virtual Work: Free Body Diagram

$$U = mg[\ell\cos\theta + \frac{\ell}{2}\cos\varphi]$$

$$\delta U = mg\ell[-\sin\theta\delta\theta - \frac{1}{2}\sin\varphi\delta\varphi] \equiv Q_m \delta q^m$$

$$\{Q_m\} = \begin{Bmatrix} Q_\theta \\ Q_\varphi \end{Bmatrix} = \begin{Bmatrix} -mg\ell\sin\theta \\ -\frac{1}{2}mg\ell\sin\varphi \end{Bmatrix}$$

4. Apply the Lagrangian Equation of Motion:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^m}\right) - \frac{\partial T}{\partial q^m} = Q_m \, ; \quad m = 1,2$$

$$For \quad q^1 = \theta \Rightarrow$$

$$\frac{\partial T}{\partial \dot{\theta}} = m\ell^2 \dot{\theta} + \frac{1}{2} m\ell^2 \dot{\varphi} \cos(\varphi - \theta)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = m\ell^2 \ddot{\theta} + \frac{1}{2} m\ell^2 \ddot{\varphi} \cos(\varphi - \theta) + \frac{1}{2} m\ell^2 \dot{\varphi}(\dot{\varphi} - \dot{\theta})[-\sin(\varphi - \theta)]$$

$$\frac{\partial T}{\partial \theta} = \frac{1}{2} m\ell^2 \dot{\theta}\dot{\varphi} \sin(\varphi - \theta)$$

Substituting into the Lagrange's equation, we get:

$$m\ell^2 \ddot{\theta} + \frac{1}{2} m\ell^2 \ddot{\varphi} \cos(\varphi - \theta) + \frac{1}{2} m\ell^2 \dot{\varphi}^2 \sin(\varphi - \theta) = -mg\ell \sin\theta$$

$For \quad q^2 = \varphi$, **we can similarly solve for the second equation of motion.**

Example: Express Differential Equation of Motion of the YOYO shown (an example for *Rheonomic* System where $x_i = f_i(q^m, t)$)?

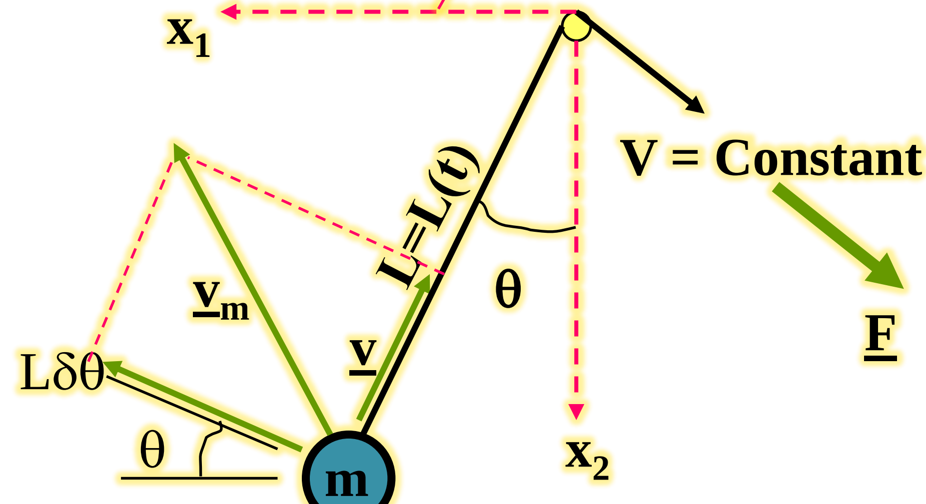
Consider the Pendulum as a 1-DOF system described by "θ". Then "$\underline{F}$" (tensile force) does no virtual work, because "L" does not change in virtual displacement, which holds time constant.

0 = (By Virtual Concept)

$$L = L_0 - vt \quad \Rightarrow \dot{L} = -v \quad \Rightarrow \delta L = -v\delta t = 0$$

$$\delta U = F.\delta L = 0$$

1. Motion:



Initial Length is; L_0
Current Length is; $L=L_0-vt$

$$\underline{r} = L\sin\theta\underline{e}_1 + L\cos\theta\underline{e}_2 =$$

$$= \underbrace{(L_0 - vt)\sin\theta}_{x_1}\underline{e}_1 + \underbrace{(L_0 - vt)\cos\theta}_{x_2}\underline{e}_2$$

Note that: $x_i = f_i(\theta, t)$

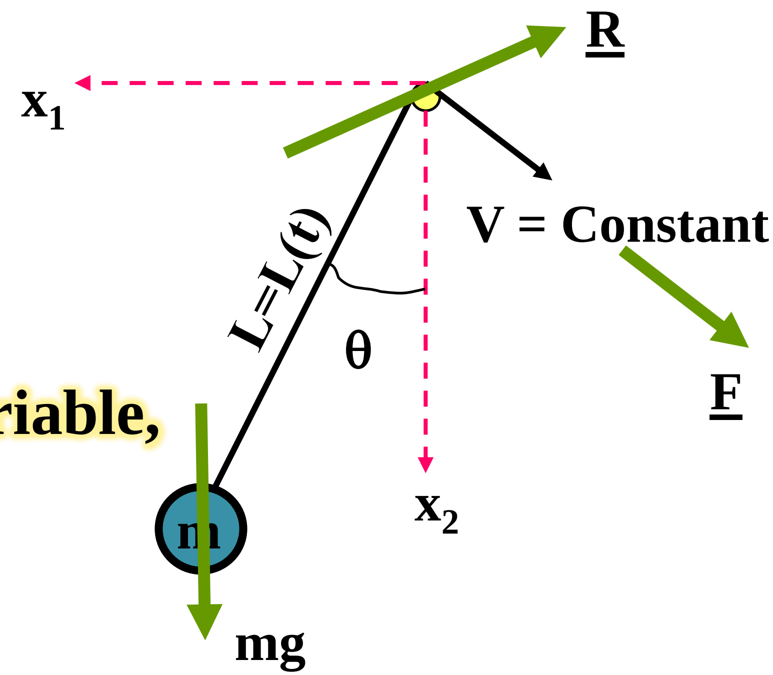
$$\dot{\underline{r}} = L\dot{\theta}\cos\theta\underline{e}_1 - L\dot{\theta}\sin\theta\underline{e}_2 - v\sin\theta\underline{e}_1 - v\cos\theta\underline{e}_2$$

$$(v_m)^2 = L^2\dot{\theta}^2 + v^2$$

2. Kinetic Energy:

$$T = \frac{1}{2}mv_m^2 = \frac{1}{2}m(L^2\dot{\theta}^2 + v^2)$$

3. Virtual Work: θ is the only variable, and Free Body Diagram;



$$\begin{cases} \delta U = -mgL\sin\theta\delta\theta \Rightarrow \\ Q_\theta = -mgL\sin\theta \end{cases}$$

4. Apply Lagrange's Equation of Motion:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = Q_\theta$$

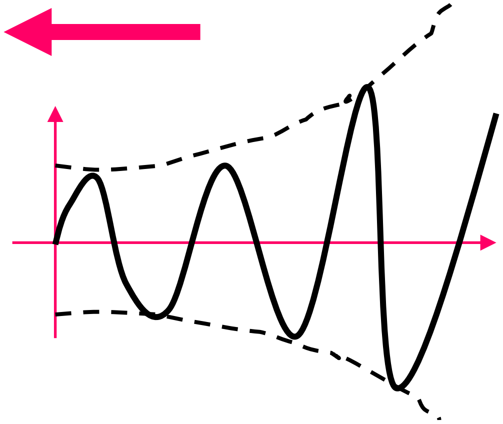
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = \frac{d}{dt}(mL^2\dot{\theta}) = mL^2\ddot{\theta} + 2mL\dot{L}\dot{\theta} = mL^2\ddot{\theta} - 2mLv\dot{\theta}$$

$$\frac{\partial T}{\partial \theta} = 0$$

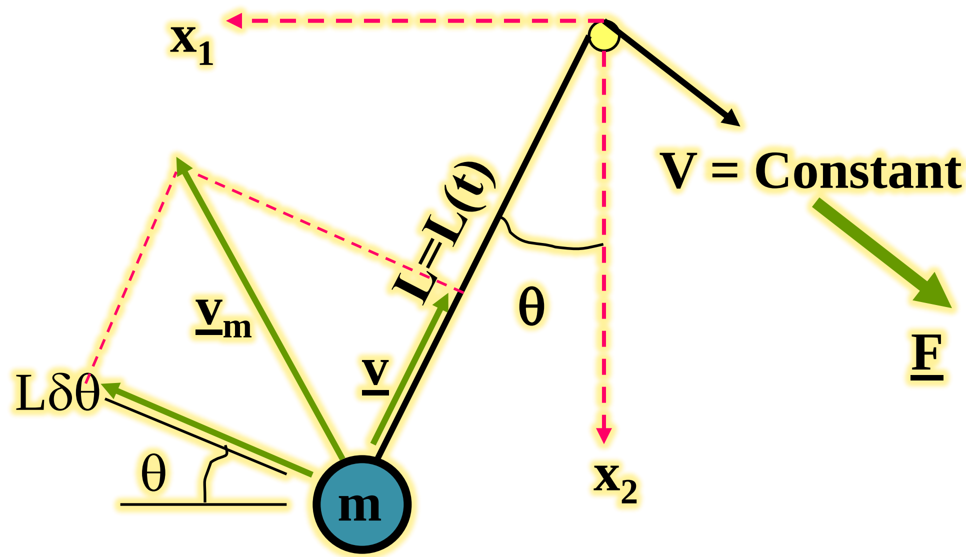
$$mL^2\ddot{\theta} - 2mLv\dot{\theta} + mgL\sin\theta = 0$$

$$\ddot{\theta} - \frac{2v}{L}\dot{\theta} + \frac{g}{L}\theta = 0$$ **(For Small Angles)**

(Negative damping results in an Unstable system)



Another Approach to the YOYO Problem: **“θ” and “L” are constrained generalized coordinates that satisfy L=L(t). Then, “L” increases by “δL” in a virtual movement, and “F” does virtual work. Then, we need to formulate the problem by Lagrange’s Equation for Dependent Set of Generalized Coordinates.**



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