بسم الله الرحمن الرحيم

By: Professor Ali Meghdari

ENERGY PRINCIPLES

***Purpose*:**

- **To Study *Energy Principles* of Dynamics.**

***Topics*:**

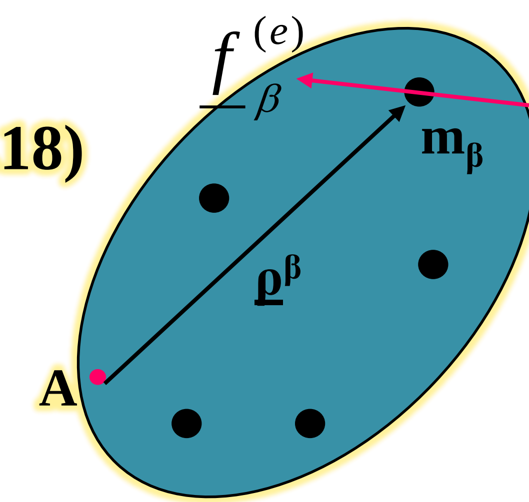
- **Kinetic Energy**
- **Work**
- **Leibniz (Leibnutz) Equation of Motion**
- **Conservative Force Field**

Work: The work done for individual paths of particles:

$$\dot{U} = \dot{U}^{(e)} = \sum_{\beta=1}^{N} \underline{v}^{\beta} \cdot \underline{f}_{\beta}^{(e)} = \sum_{\beta=1}^{N} \underline{v}^{\beta} \cdot \underline{f}_{\beta} \qquad \textbf{(10.14)}$$

Theorem-41: **For a Rigid Body, the work done can be equated to the sum of the work done by the resultant force in displacing any convenient point "A" in the rigid body and that done by the resultant moment about the same point in rotating the rigid body.**

$$\dot{U} = \underline{v}^{A} \cdot \underline{f}_{R} + \underline{\omega} \cdot \underline{M}^{A} \qquad \textbf{(10.18)}$$



Virtual Work

Definition: ***Virtual Displacement*** **of a system is hypothetical but admissible in accordance with the mechanical constraints.Its value is infinitesimal but arbitrary and definitely independent of time.**

In a virtual movement, generalized coordinates of the system are considered to be incremented by infinitesimal amounts "δq_i" from the values they have at an arbitrary instant with time held constant.

a) For individual paths of particles:

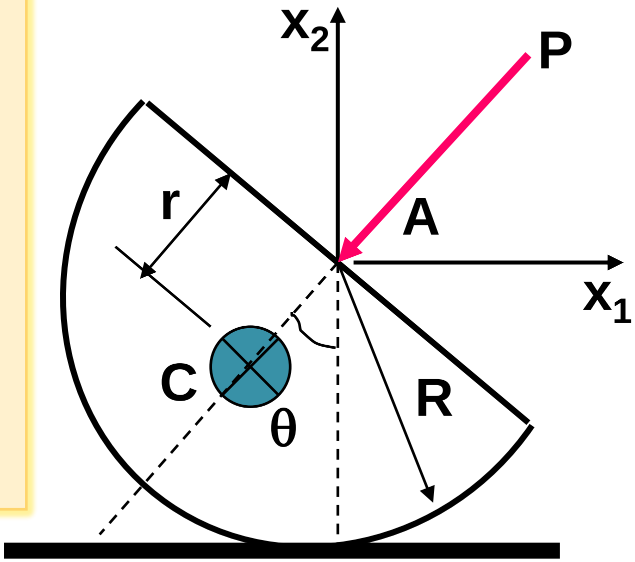
$$\delta U = \sum_{\beta=1}^{N} \underline{f}_{\beta} \cdot \delta \underline{r}^{\beta} \qquad (10.22)$$

b) For a Rigid Body:

$$\delta U = \underline{f}_R \cdot \delta \underline{r}^A + \underline{M}^A \cdot \delta \underline{\theta} \qquad (10.23)$$

δ: Virtual (Imaginary) Notation

Example: A half cylinder of mass "m" rocks without slipping on a flat level surface subjected to a force that is perpendicular to its diametrical plane at all time. What is the virtual work in the coordinate of angular position?



$$\delta U = \underline{f}_R \cdot \delta \underline{r}^A + \underline{M}^A \cdot \delta \underline{\theta}$$

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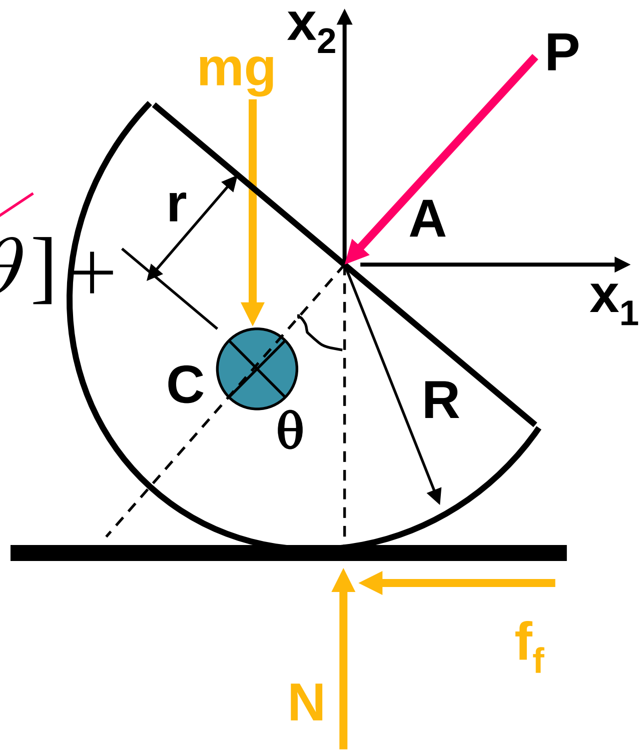
$$\underline{f}_R = -mg\underline{e}_2 + P(-\sin\theta \underline{e}_1 - \cos\theta \underline{e}_2) + N\underline{e}_2 - f_f \underline{e}_1$$

$$\delta \underline{r}^A = R\delta\theta \underline{e}_1, \quad \delta\underline{\theta} = -\delta\theta \underline{e}_3$$

$$\underline{M}^A = (-f_f R + mgr\sin\theta)\underline{e}_3$$

$$\delta U = [(-P\sin\theta)R\delta\theta - \cancel{f_f R\delta\theta}] +$$

$$+[\cancel{f_f R\delta\theta} - mgr\sin\theta\delta\theta]$$



$$\delta U = -(mgr + PR)\sin\theta\delta\theta$$

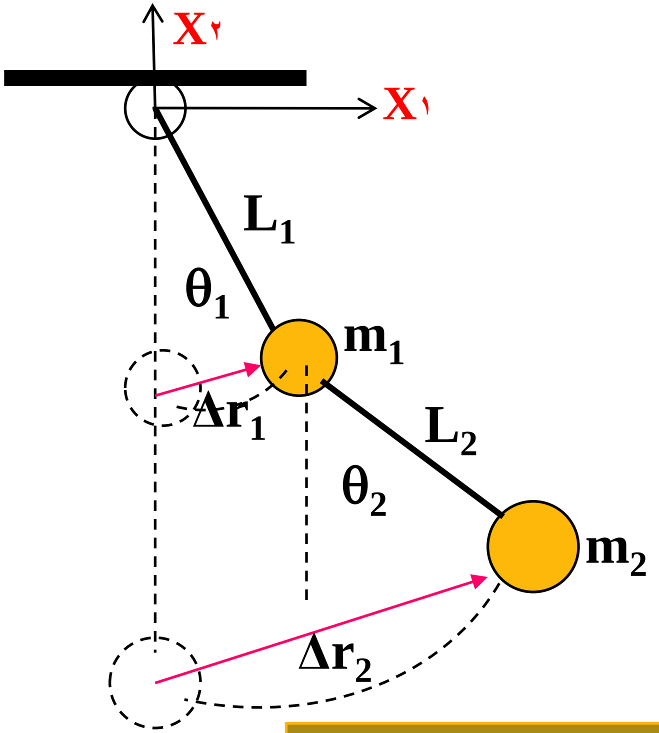
(f_f and N: none-working forces)

Leibniz Equation of Motion

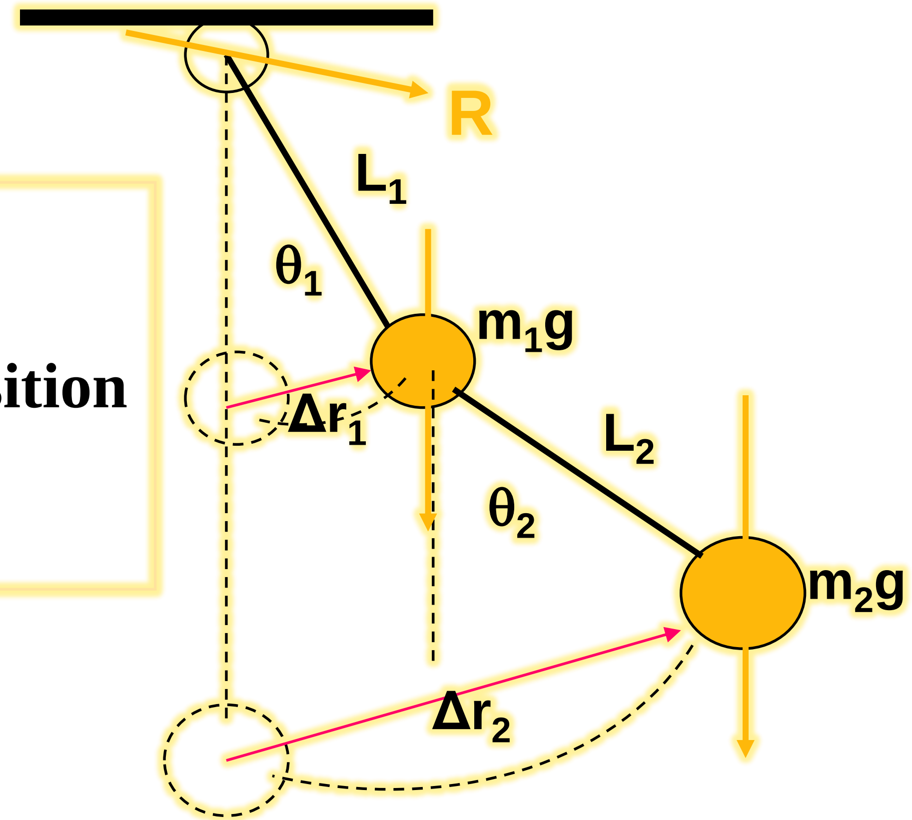
***Leibniz Principle*: The change of kinetic energy of a system is equal to the work done on the system.**

$$U = \Delta T\,, \quad dU = dT\,, \quad \dot{U} = \dot{T}\,, \quad \delta U = \delta T \qquad \textbf{(10.23)}$$

Example: **A double pendulum of given masses (m_1 and m_2) and given lightweight rigid links (L_1 and L_2) is set in motion. Determine the equations of motion?**



The work done by two gravitational forces from the fixed equilibrium position to the arbitrary position (θ_1 ,θ_2) is:



$$U = -m_1 g L_1 (1-\cos\theta_1) - m_2 g[(L_1(1-\cos\theta_1) + L_2(1-\cos\theta_2)]$$

$$\delta U = -(m_1+m_2) g L_1 \sin\theta_1 \delta\theta_1 - m_2 g L_2 \sin\theta_2 \delta\theta_2$$

$$(Virtual \quad Symbol \equiv Differencial \quad Operator)$$

Computation of Kinetic Energy:

$$\underline{V}_1 = L_1\dot{\theta}_1(\cos\theta_1\underline{e}_1 + \sin\theta_1\underline{e}_2)$$

$$\underline{V}_2 = L_1\dot{\theta}_1(\cos\theta_1\underline{e}_1 + \sin\theta_1\underline{e}_2) + L_2\dot{\theta}_2(\cos\theta_2\underline{e}_1 + \sin\theta_2\underline{e}_2)$$

$$V_1^2 = L_1^2\dot{\theta}_1^2$$

$$V_2^2 = L_1^2\dot{\theta}_1^2 + L_2^2\dot{\theta}_2^2 + 2L_1L_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_2 - \theta_1)$$

$$T = \frac{1}{2}m_1V_1^2 + \frac{1}{2}m_2V_2^2 =$$

$$T = \frac{1}{2}[(m_1 + m_2)L_1^2\dot{\theta}_1^2 + m_2L_2^2\dot{\theta}_2^2 + 2m_2L_1L_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_2 - \theta_1)]$$

$$\dot{T} = [(m_1 + m_2)L_1^2\ddot{\theta}_1 + m_2 L_1 L_2 \ddot{\theta}_2 \cos(\theta_2 - \theta_1) + m_2 L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_2 - \theta_1)]\dot{\theta}_1 +$$
$$+ [m_2 L_2^2 \ddot{\theta}_2 + m_2 L_1 L_2 \ddot{\theta}_1 \cos(\theta_2 - \theta_1) - m_2 L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_2 - \theta_1)]\dot{\theta}_2$$

For small θ_1 and θ_2 , the sinusoidal functions are first expanded in power series. Then, when $(\theta, \dot{\theta}, \ddot{\theta})$ are considered of the same order and terms of order $O(\theta^2)$ in the brackets are omitted, we have:

$$\dot{T} = [(m_1 + m_2)L_1^2\ddot{\theta}_1 + m_2 L_1 L_2 \ddot{\theta}_2]\dot{\theta}_1 +$$
$$+ [m_2 L_2^2 \ddot{\theta}_2 + m_2 L_1 L_2 \ddot{\theta}_1]\dot{\theta}_2$$

Then, the Virtual change in Kinetic Energy is:

$$\delta T = [(m_1 + m_2)L_1^2\ddot{\theta}_1 + m_2 L_1 L_2 \ddot{\theta}_2]\delta\theta_1 +$$
$$+ [m_2 L_2^2 \ddot{\theta}_2 + m_2 L_1 L_2 \ddot{\theta}_1]\delta\theta_2$$

Applying the Leibniz Principle we have:

$$\delta T = \delta U$$

$$[(m_1+m_2)L_1^2\ddot{\theta}_1 + m_2 L_1 L_2 \ddot{\theta}_2 + (m_1+m_2)gL_1\theta_1]\delta\theta_1 +$$
$$+[m_2 L_2^2\ddot{\theta}_2 + m_2 L_1 L_2 \ddot{\theta}_1 + m_2 g L_2 \theta_2]\delta\theta_2 = 0$$

Since θ_1 and θ_2 are independent coordinates, $\delta\theta_1$ and $\delta\theta_2$ are independent and arbitrary. Therefore, the coefficients of $\delta\theta_i$ must vanish:

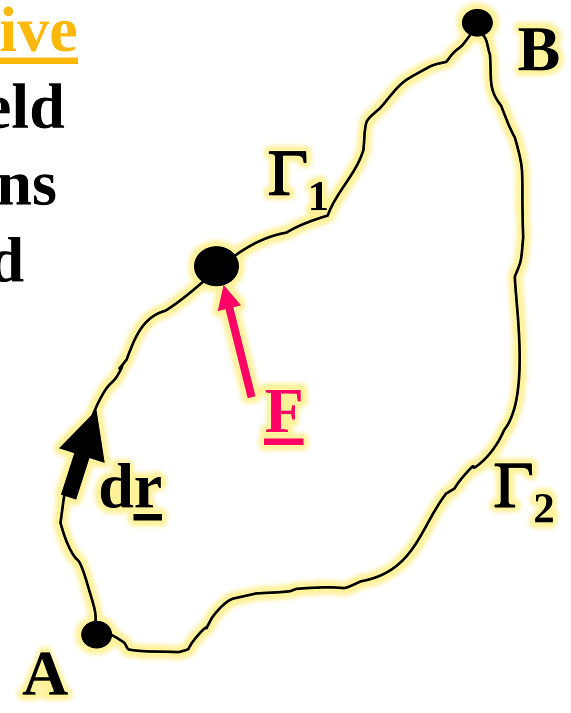
$$\begin{cases} (m_1+m_2)L_1^2\ddot{\theta}_1 + m_2 L_1 L_2 \ddot{\theta}_2 + (m_1+m_2)gL_1\theta_1 = 0 \\ m_2 L_2^2\ddot{\theta}_2 + m_2 L_1 L_2 \ddot{\theta}_1 + m_2 g L_2 \theta_2 = 0 \end{cases}$$

D.E.M.

Conservative Force Field

In computing work, it is noticed that some of the work integral depends only on the initial and final positions of the path. Yet some are path or time dependent, perhaps even both.

The *Force Field* is said to be conservative if the work done on a particle in the field depends on the initial and final positions only. Otherwise, the *Force Field* is said to be non-conservative.

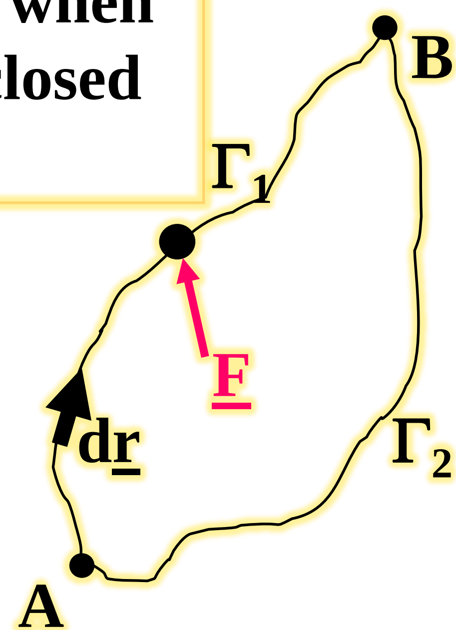


In order for the work integral to depend only on the limits of integration (i.e., to be independent of the path taken from A to B) it is necessary and sufficient that "$\underline{F}.d\underline{r}$" be an exact differential, which is denoted by "dU" .

$$\underline{F}.d\underline{r} = dU \qquad (10.24)$$

Definition: **A Force Field "$\underline{F}$" is conservative when the work done on a particle which travels a closed path in the field is zero.**

$$\oint_{\Gamma} \underline{F} \cdot d\underline{r} = 0 \qquad (10.25)$$



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Theorem-42: **In order for "F.dr" in Equation (10.24) to be an exact differential "dU", it is necessary and sufficient that $\underline{\nabla} \times \underline{F} = 0$. Or, in other words, a force "$\underline{F}$" is said to be conservative if and only if $\underline{\nabla} \times \underline{F} = 0$. (Curl of $\underline{F}$ =0)**

Proof: **1. Consider "F.dr" to be an exact differential. Then working in rectangular coordinates, we may write:**

$$\underline{F} \cdot d\underline{r} = dU = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz = (\underline{\nabla}U).d\underline{r}$$

(10.26)

$$where: \underline{\nabla} = \frac{\partial}{\partial x}\underline{i} + \frac{\partial}{\partial y}\underline{j} + \frac{\partial}{\partial z}\underline{k}. \qquad \Rightarrow$$

$$\underline{F} = \underline{\nabla}U = grad(U) \qquad (10.27)$$

$$\underline{F} = \underline{\nabla} U = grad(U) \qquad (10.27)$$

$$F_x = \frac{\partial U}{\partial x}, F_y = \frac{\partial U}{\partial y}, F_z = \frac{\partial U}{\partial z}$$

(Rectangular components of the force)

We thus conclude that for a conservative system, the force is expressible as the gradient of a scalar potential function, "U". Therefore, the force is a function of spatial coordinates only.

Finally, we can conclude from Eq. (10.27) that if $\underline{F}$ is conservative, (i.e., if $\underline{F}\cdot d\underline{r} = dU$ is a perfect differential) then:

$$\underline{\nabla} \times \underline{F} = (\underline{\nabla} \times \underline{\nabla})U = 0 \qquad (10.28)$$

2. Conversely, consider that for a particular force "$\underline{F}$" it has been shown that "$\underline{\nabla} \times \underline{F} = 0$".

> **Definition: The Potential Energy "V" is a scalar function which is effectively the work one would perform to transport the mechanical system from a *datum* state to the existing state while holding the system in equilibrium with the external force and internal resilience.**

$$dV = -dU \Rightarrow V_2 - V_1 = -\int_1^2 \underline{F} \cdot d\,\underline{r} \qquad \textbf{(10.29)}$$

Example: **Given a force system characterized by $F_x = 2xy + y$ and $F_y = x^2 - y^2 + x$, is this a conservative system?**

If this is a conservative system, then:
which yields the requirement that;

$$\underline{\nabla} \times \underline{F} = 0$$

$$\left(\frac{\partial}{\partial x}\underline{i} + \frac{\partial}{\partial y}\underline{j}\right) \times \left(F_x \underline{i} + F_y \underline{j}\right) = 0 \Rightarrow$$

$$\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = 0 \qquad \Rightarrow$$

$$\frac{\partial}{\partial x}(x^2 - y^2 + x) - \frac{\partial}{\partial y}(2xy + y) = 2x + 1 - 2x - 1 = 0$$

متشکرم