بسم الله الرحمن الرحیم

By: Professor Ali Meghdari

ENERGY PRINCIPLES

***Purpose*:**

- **To Study *Energy Principles* of Dynamics.**

***Topics*:**

- **Kinetic Energy**
- **Work**
- **Leibniz (Leibnutz) Equation of Motion**
- **Conservative Force Field**

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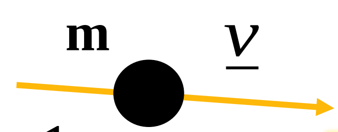
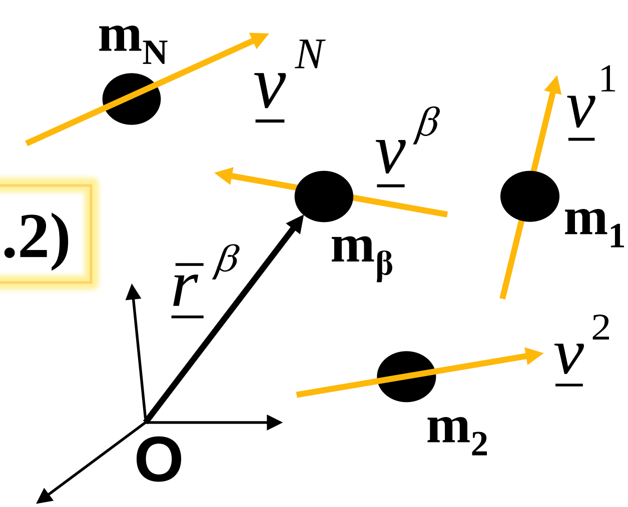
Kinetic Energy (K.E.): For a single particle is defined by its inertia property and speed.

K.E. for a Single Particle:



$$T=\frac{1}{2}mv^2 \quad or \quad T=\frac{1}{2}m\underline{v}\cdot\underline{v}=\frac{1}{2}m\dot{x}_i\dot{x}_i \qquad \textbf{(10.1)}$$

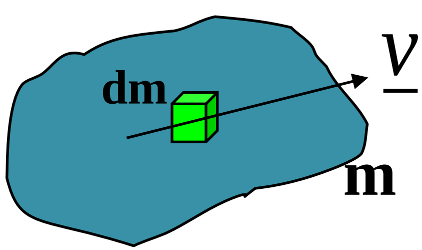
K.E. for a System of Particles:



$$T=\sum_{\beta=1}^{N}T_\beta=\frac{1}{2}\sum_{\beta=1}^{N}m_\beta v_\beta^2=\frac{1}{2}\sum_{\beta=1}^{N}m_\beta\underline{v}^\beta\cdot\underline{v}^\beta \qquad \textbf{(10.2)}$$

K.E. for a Continuum:

$$T=\frac{1}{2}\int_m\underline{v}\cdot\underline{v}dm \qquad \textbf{(10.3)}$$



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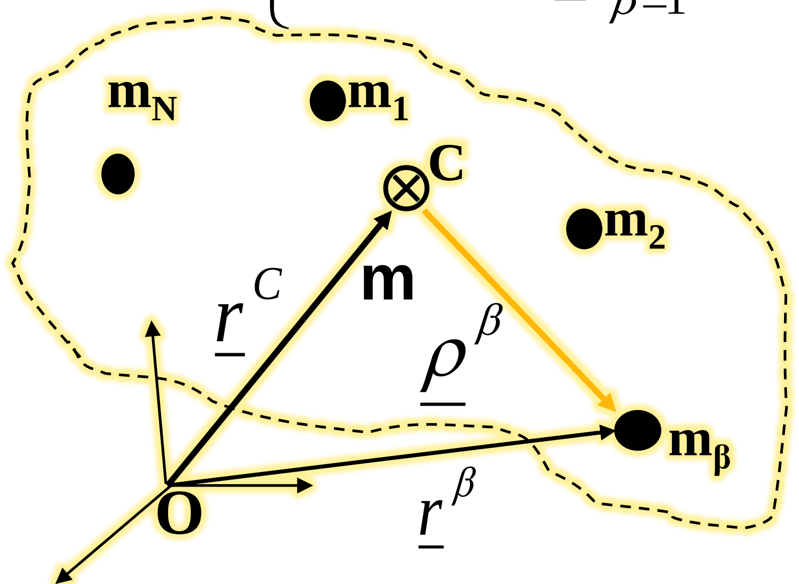
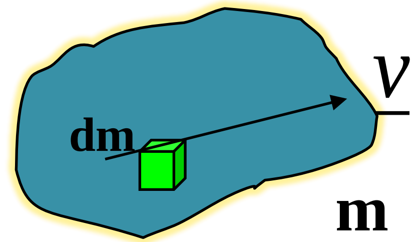
Theorem-34: **The Total K.E. of a constant mass system is equal to the K.E. due to the *equivalent mass* at the mass center (*K.E. due to motion of the mass center*) plus the K.E. due to the motion of *individual particles* with respect to the mass center.**

$$T = T^{(e)} + T^{(i)} \Rightarrow \begin{cases} T^{(e)} = \dfrac{1}{2} m v_C^2 = \dfrac{1}{2} m \underline{v}^C \cdot \underline{v}^C \\ T^{(i)} = \dfrac{1}{2} \sum_{\beta=1}^{N} m_\beta \dot{\underline{\rho}}^\beta \cdot \dot{\underline{\rho}}^\beta \end{cases} \qquad \textbf{(10.4)}$$



for a *Continuum*: $T^{(i)} = \dfrac{1}{2}\int_m \underline{\dot{\rho}} \cdot \underline{\dot{\rho}}\, dm$



Proof:

$$T = \frac{1}{2}\sum_{\beta=1}^{N} m_\beta \underline{v}^\beta \cdot \underline{v}^\beta = \frac{1}{2}\sum_{\beta=1}^{N} m_\beta (\underline{v}^C + \underline{\dot{\rho}}^\beta)\cdot(\underline{v}^C + \underline{\dot{\rho}}^\beta) =$$

$$= \frac{1}{2}\sum_{\beta=1}^{N} m_\beta \underline{v}^C \cdot \underline{v}^C + \frac{1}{2}\sum_{\beta=1}^{N} m_\beta \underline{\dot{\rho}}^\beta \cdot \underline{\dot{\rho}}^\beta + \underbrace{\sum_{\beta=1}^{N} m_\beta \underline{\dot{\rho}}^\beta \cdot \underline{v}^C}$$

$$\equiv \Big(\frac{d}{dt}\underbrace{\sum_{\beta=1}^{N} m_\beta \underline{\rho}^\beta}\Big) \cdot \underline{v}^C$$

(The first moment of a mass system about its mass center is zero)

$$T = \frac{1}{2}\sum_{\beta=1}^{N} m_\beta \underline{v}^C \cdot \underline{v}^C + \frac{1}{2}\sum_{\beta=1}^{N} m_\beta \underline{\dot{\rho}}^\beta \cdot \underline{\dot{\rho}}^\beta \qquad \textbf{(10.4)}$$

Kinetic Energy of a Rigid Body:

Theorem-35: **For a Rigid System, the expression for $T^{(e)}$ is the same as before (*K.E. due to motion of the mass center*), and $T^{(i)}$ is completely defined by the Central Inertia Tensor "I_{ij}^{C}" and the angular velocity vector "$\underline{\omega}$" of the Rigid Body.**

$$T = T^{(e)} + T^{(i)} \Rightarrow \begin{cases} T^{(e)} = \dfrac{1}{2} m \underline{v}^{C} \cdot \underline{v}^{C} = \dfrac{1}{2} \underline{v}^{C} \cdot \underline{P} \\ T^{(i)} = \dfrac{1}{2} I_{ij}^{C} \omega_i \omega_j = \dfrac{1}{2} \underline{\omega} \cdot \underline{\underline{I}}^{C} \cdot \underline{\omega} \end{cases}$$

(10.5)

Proof:

$$T^{(i)} = \frac{1}{2}\int_m \underline{\dot{\rho}} \cdot \underline{\dot{\rho}}\,dm = \frac{1}{2}\int_m (\underline{\omega} \times \underline{\rho}) \cdot (\underline{\omega} \times \underline{\rho})\,dm =$$

$$= \frac{1}{2}\int_m \underline{\omega} \cdot \underline{\rho} \times (\underline{\omega} \times \underline{\rho})\,dm = \frac{1}{2}\underline{\omega} \cdot \underbrace{\int_m \underline{\rho} \times (\underline{\omega} \times \underline{\rho})\,dm}_{\underline{H}^C} =$$

$$= \frac{1}{2}\underline{\omega} \cdot \underline{H}^C = \frac{1}{2}\underline{\omega} \cdot \underline{\underline{I}}^C \cdot \underline{\omega}$$

Since:

$$\underline{\dot{\rho}} = \underline{\omega} \times \underline{\rho} \quad , \quad \underline{\omega} = \underline{\omega}_{RigidBody}, and$$

$$\underline{A} \times \underline{B} \cdot \underline{C} = \underline{A} \cdot \underline{B} \times \underline{C}$$

Hence:

$$\begin{cases} T = \dfrac{1}{2} m \underline{v}^{C} \cdot \underline{v}^{C} + \dfrac{1}{2} \underline{\omega} \cdot \underline{\underline{I}}^{C} \cdot \underline{\omega} \\ or \\ T = \dfrac{1}{2} \underline{v}^{C} \cdot \underline{P} + \dfrac{1}{2} \underline{\omega} \cdot \underline{H}^{C} \end{cases} \qquad \textbf{(10.5)}$$

In terms of Principal Coordinates at "C":

$$T^{(i)} = \frac{1}{2}\left(I_1^C \omega_1^2 + I_2^C \omega_2^2 + I_3^C \omega_3^2\right) \qquad \textbf{(10.6)}$$

***Theorem-36*:** **For a Rigid System, when there is one zero velocity body point A (*a fixed point or momentarily has zero velocity*), the *Total Kinetic Energy* can be readily obtained by referring to that point as the moment center.**

$$T = \frac{1}{2} I_{ij}^{A} \omega_i \omega_j = \frac{1}{2} \underline{\omega} \cdot \underline{\underline{I}}^{A} \cdot \underline{\omega} = \frac{1}{2} \underline{\omega} \cdot \underline{H}^{A} \qquad \textbf{(10.7)}$$

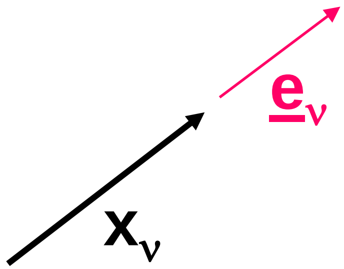
In terms of Principal Coordinates at "A" :

$$T = \frac{1}{2} \left(I_1^A \omega_1^2 + I_2^A \omega_2^2 + I_3^A \omega_3^2 \right) \qquad \textbf{(10.8)}$$

K.E. with reference to the ***Instantaneous Axis of Rotation*** of the Rigid Body:

Let $\{x_i\}$ be chosen such that x_ν is the Instantaneous Axis of Rotation; then:

$$\underline{\omega} = \omega \underline{e}_\upsilon \qquad or \qquad \omega_i = \omega \ell_{\upsilon i}$$



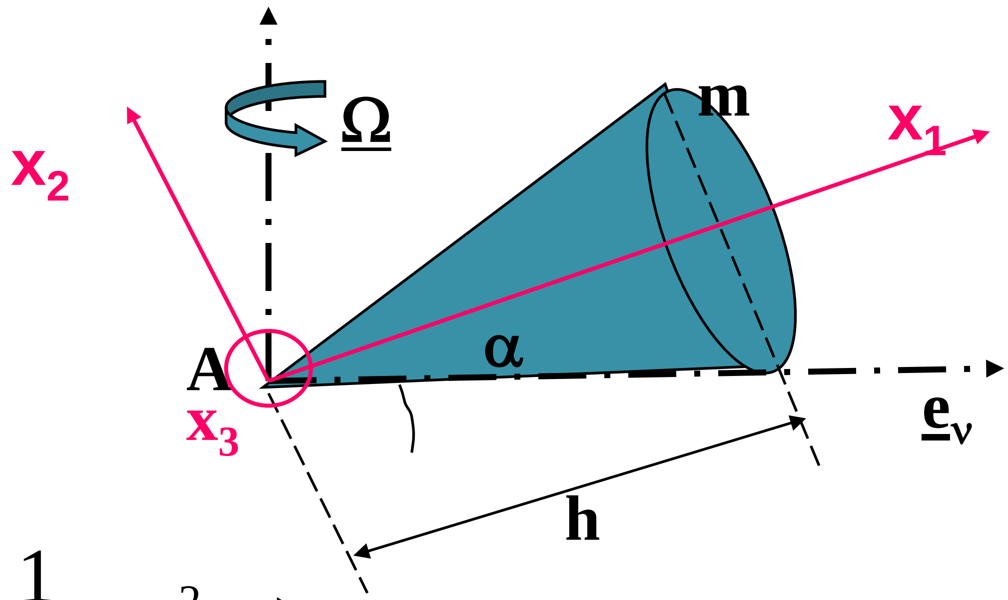
$$T = \frac{1}{2} I_{ij}^{A} \ell_{\upsilon i} \ell_{\upsilon j} \omega^2 = \frac{1}{2} I_{\upsilon\upsilon}^{A} \omega^2 \qquad \textbf{(10.9)}$$

And for a *Rigid Body in Plane Motion*:

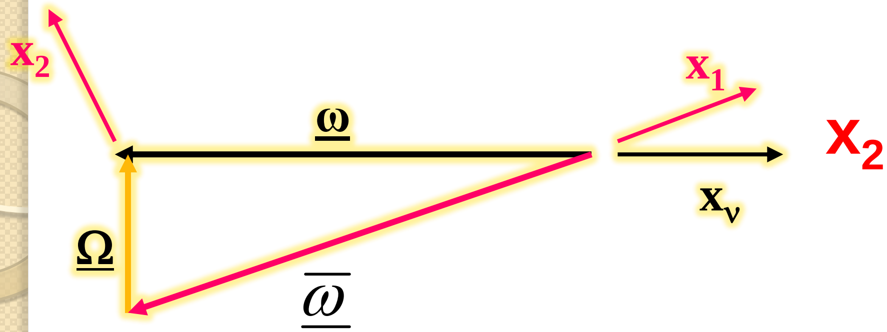
$$T = T^{(i)} = \frac{1}{2} I_{\upsilon\upsilon}^{C} \omega^2 \qquad \textbf{(10.10)}$$

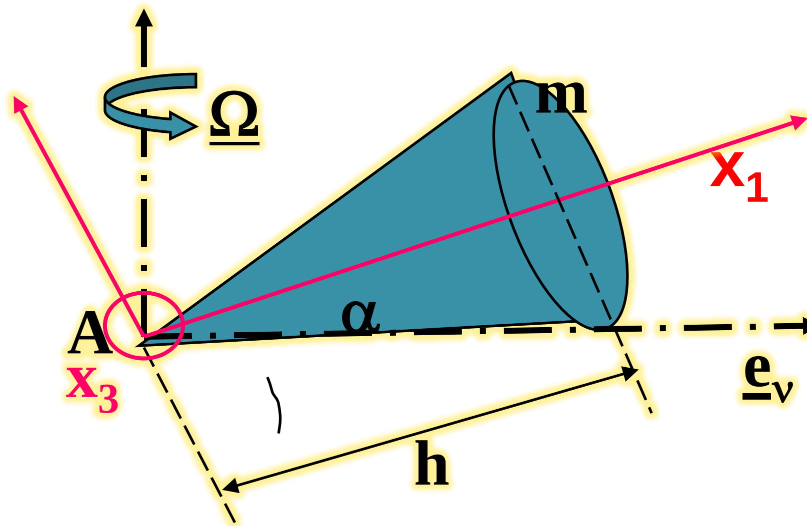
Example: A conical shell of height "h", cone angle "α" and mass "m" rolls without slipping on a horizontal plane about a vertical axis. If its angular velocity about the vertical axis is "Ω", determine the Kinetic Energy of the conical shell?

Method I: Use Principal Coordinates at "A".



$$\begin{cases} I_1^A = \frac{1}{2} m h^2 \tan^2 \alpha \\ I_2^A = I_3^A = \frac{1}{2} m h^2 (1 + \frac{1}{2} \tan^2 \alpha) \end{cases}$$





$$\begin{cases} \underline{\omega} = \omega(-\cos\alpha \underline{u}_1 + \sin\alpha \underline{u}_2) \\ \cot g\alpha = \dfrac{\omega}{\Omega} \Rightarrow \omega = \Omega \cot g\alpha \\ \underline{\omega} = \Omega \cot g\alpha(-\cos\alpha \underline{u}_1 + \sin\alpha \underline{u}_2) \end{cases}$$

(Absolute velocity along the contact line)

$$T = \frac{1}{2}[I_1^A \omega_1^2 + I_2^A \omega_2^2 + I_3^A \omega_3^2] =$$

$$= \frac{1}{4} m h^2 \Omega^2 [2\cos^2\alpha + \frac{1}{2}\sin^2\alpha]$$

Method II: Use Instantaneous Axis of Rotation.
By transform equation of inertia tensor, we have:

$$I_{\upsilon\upsilon}^{A} = I_{ij}^{A}\ell_{\upsilon i}\ell_{\upsilon j} = I_{1}^{A}\ell_{\upsilon 1}^{2} + I_{2}^{A}\ell_{\upsilon 2}^{2} + I_{3}^{A}\ell_{\upsilon 3}^{2} =$$

$$= \frac{1}{2}mh^{2}[2\sin^{2}\alpha + \frac{1}{2}\tan^{2}\alpha\sin^{2}\alpha]$$

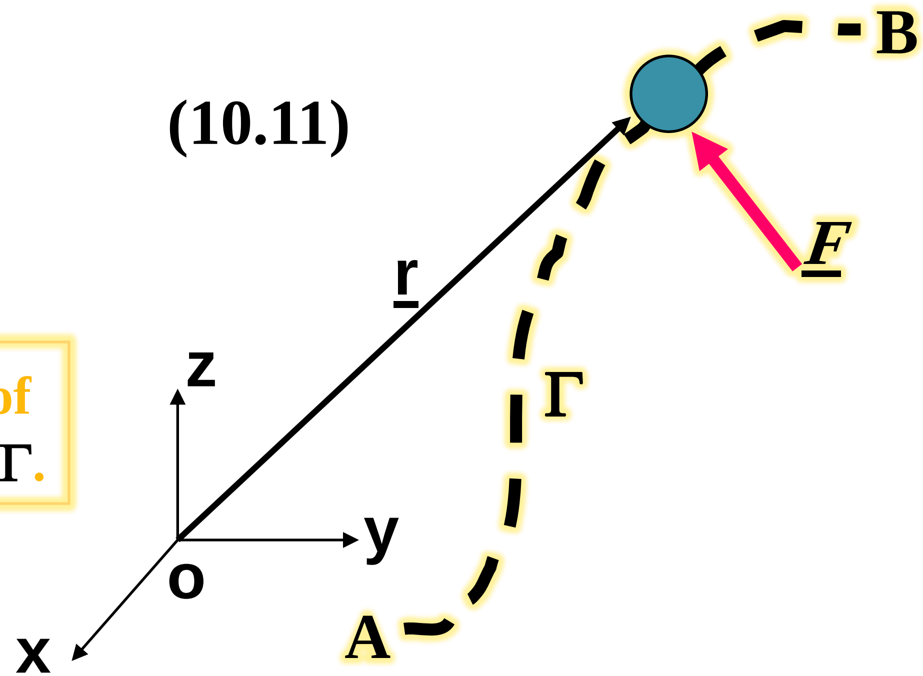
$$T = \frac{1}{2}I_{\upsilon\upsilon}^{A}\omega^{2} = \frac{1}{4}mh^{2}\Omega^{2}\cot g^{2}\alpha[2\sin^{2}\alpha + \frac{1}{2}\tan^{2}\alpha\sin^{2}\alpha] =$$

$$= \frac{1}{4}mh^{2}\Omega^{2}[2\cos^{2}\alpha + \frac{1}{2}\sin^{2}\alpha]$$

Work: Work applied on a system depends on both the force field and the displacement of the system in the field.

Definition: The work done by a force on a particle as it travels along the path Γ is defined as:

$$U = \int_{\Gamma} \underline{f} \cdot d\underline{r} \qquad (10.11)$$



$d\underline{r}$: Differential displacement of the particle along the path Γ.

Differential Expression of Work:

$$dU = \underline{f} \cdot d\underline{r}$$

Power: Time rate of doing Work:

$$\dot{U} = \underline{f} \cdot \underline{v}$$

(10.12)

***Theorem-37*: For a system of particles, while the totality of the internal forces and that of internal moments of force do vanish, the internal forces do contribute to the total work done on the system.**

***Internal Work*: The work done by internal forces on the system.**

$$\left\{ \sum_{\beta=1}^{N} \underline{f}_{\beta}^{(i)} = 0, \quad \sum_{\beta=1}^{N} \underline{r}^{\beta} \times \underline{f}_{\beta}^{(i)} = 0 \right\} \quad , \dot{U}^{(i)} = \sum_{\beta=1}^{N} \underline{v}^{\beta} \cdot \underline{f}_{\beta}^{(i)}$$

(10.13)

External Work: **The work done by external forces on the system.**

$$\dot{U}^{(e)} = \sum_{\beta=1}^{N} \underline{v}^{\beta} \cdot \underline{f}_{\beta}^{(e)} \qquad \textbf{(10.14)}$$

Theorem-38: **The Total Work done on a system of particles is the combination of the external and internal works.**

$$U = U^{(i)} + U^{(e)} \qquad or \qquad \dot{U} = \dot{U}^{(i)} + \dot{U}^{(e)} \qquad \textbf{(10.15)}$$

Note: **Since the interior forces are equal, opposite and collinear pairs, they will do work only through extension and contraction of the pair of corresponding particles.**

***Theorem-39*: The value of Internal Work is independent of the reference frame.**

Proof: Choose any arbitrary reference frame with reference point “O” and angular velocity “$\underline{\Omega}$”.

$$\dot{U}^{(i)} = \sum_{\beta=1}^{N} \underline{v}^{\beta} \cdot \underline{f}_{\beta}^{(i)} = \sum_{\beta=1}^{N} [(\overbrace{\underline{\bar{v}}^{\beta} + \underline{v}^{O} + \underline{\Omega} \times \underline{\bar{r}}^{\beta}}^{\underline{v}^{\beta}}) \cdot \underline{f}_{\beta}^{(i)}] =$$

$$= \sum_{\beta=1}^{N} \underline{\bar{v}}^{\beta} \cdot \underline{f}_{\beta}^{(i)} + \underline{v}^{O} \cdot \cancelto{0}{\sum_{\beta=1}^{N} \underline{f}_{\beta}^{(i)}} + \underbrace{\sum_{\beta=1}^{N} \underline{\Omega} \times \underline{\bar{r}}^{\beta} \cdot \underline{f}_{\beta}^{(i)}}$$

$$= \sum_{\beta=1}^{N} \underline{\Omega} \cdot \underline{\bar{r}}^{\beta} \times \underline{f}_{\beta}^{(i)} =$$

$$= \underline{\Omega} \cdot (\cancelto{0}{\sum_{\beta=1}^{N} \underline{\bar{r}}^{\beta} \times \underline{f}_{\beta}^{(i)}})$$

$$\dot{U}^{(i)} = \sum_{\beta=1}^{N} \underline{v}^{\beta} \cdot \underline{f}_{\beta}^{(i)} = \underbrace{\sum_{\beta=1}^{N} \underline{\bar{v}}^{\beta} \cdot \underline{f}_{\beta}^{(i)}}_{} \qquad \textbf{(10.16)}$$

(Internal work as observed in the arbitrary reference frame, since the internal forces are independent of the reference frame.)

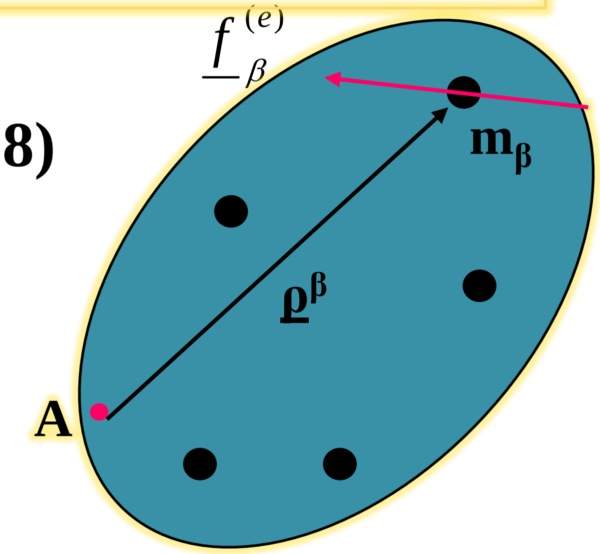
***Rigid Bodies*:** **Are the components of most mechanical systems. Therefore, we have:**

***Theorem-40*:** **For a Rigid Body the Internal Forces do no work. This is obvious when the reference frame is attached to a rigid body.**

$$\dot{U}^{(i)} = 0 \qquad \textbf{(10.17)}$$

Theorem-41: **For a Rigid Body, the work done can be equated to the sum of the work done by the resultant force in displacing any convenient point "A" in the rigid body and that done by the resultant moment about the same point in rotating the rigid body.**

$$\dot{U} = \underline{v}^{A} \cdot \underline{f}_{R} + \underline{\omega} \cdot \underline{M}^{A} \quad \mathbf{(10.18)}$$



Proof:

$$\dot{U} = \dot{U}^{(e)} = \sum_{\beta=1}^{N} \underline{v}^{\beta} \cdot \underline{f}_{\beta}^{(e)} =$$

$$\sum_{\beta=1}^{N} (\underline{v}^{A} + \underline{\omega} \times \underline{\rho}^{\beta}) \cdot \underline{f}_{\beta}^{(e)} = \underline{v}^{A} \cdot \sum_{\beta=1}^{N} \underline{f}_{\beta}^{(e)} + \sum_{\beta=1}^{N} \underline{\omega} \cdot \underline{\rho}^{\beta} \times \underline{f}_{\beta}^{(e)} =$$

$$= \underline{v}^{A} \cdot \underline{f}_{R}^{(e)} + \underline{\omega} \cdot \sum_{\beta=1}^{N} \underline{\rho}^{\beta} \times \underline{f}_{\beta}^{(e)} = \underline{v}^{A} \cdot \underline{f}_{R}^{(e)} + \underline{\omega} \cdot \underline{M}^{A}$$

$$\dot{U} = \underline{v}^A \cdot \underline{f}_R + \underline{\omega} \cdot \underline{M}^A$$

Note: **The most significant reference points are:**

1. The mass center, A = C
2. A fixed point, $\underline{v}^A = 0$

List of *Elementary Force Fields* for which the computation of work is usually simple.

a) *Non-Working Forces*: Forces at a fixed point or normal to the path. (i.e. Reactive forces at fixed support and normal reaction on a surface)

b) *Contact Forces*: Work is the inner product of the force and the total displacement of the particle.

$$U = \underline{f} \cdot \Delta\underline{r} = f_1\Delta x_1 + f_2\Delta x_2 + f_3\Delta x_3 \qquad (10.19)$$

c) ***Simple Spring Forces*:** (i.e. like extension spring and leaf spring).

$$U_s = -\int_{e_i}^{e_f} f_s dx \qquad (10.20)$$

(e_f: final deflection, e_i: initial deflection, f_s: spring force, where for *Linear Spring*: $f_s=kx$, and k: spring constant)

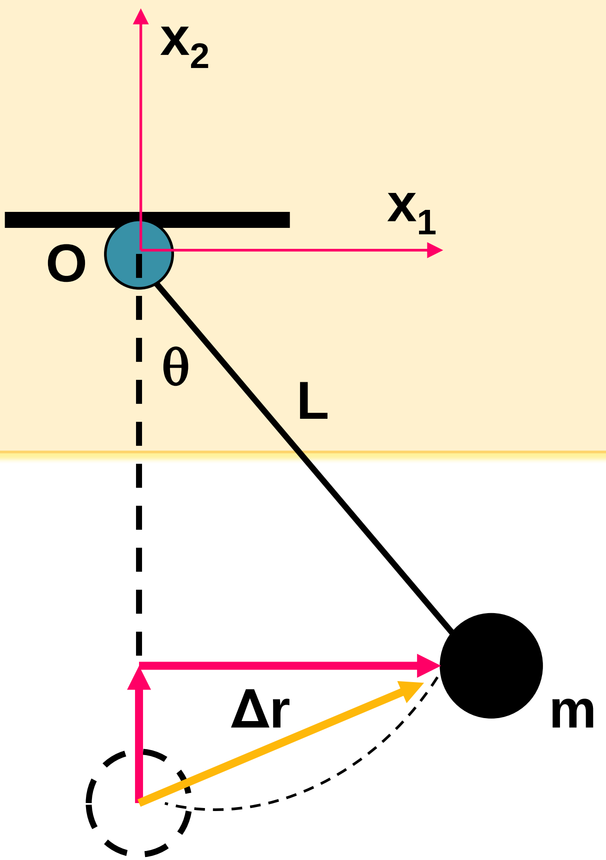
d) ***Simple Torsion Spring*:** (i.e. torsion bars and coil spring).

$$U_s = -\int_{\Delta\theta} M_s d\theta \qquad (10.21)$$

($\Delta\theta$: planar angular displacement, M_S: spring torque, where for *Linear Torsion Spring*: $M_s=k\theta$, and k: torsion spring constant)

d) ***Friction Force***: In general, work done by the friction force is path dependent. Certain friction forces such as ***Pure Rolling Friction*** does no work.

Example: A simple pendulum of mass "m" is pivoted through one end of a lightweight connecting rod "L" as shown. Determine the work done on the system in displacing from its equilibrium position to a given angular position.



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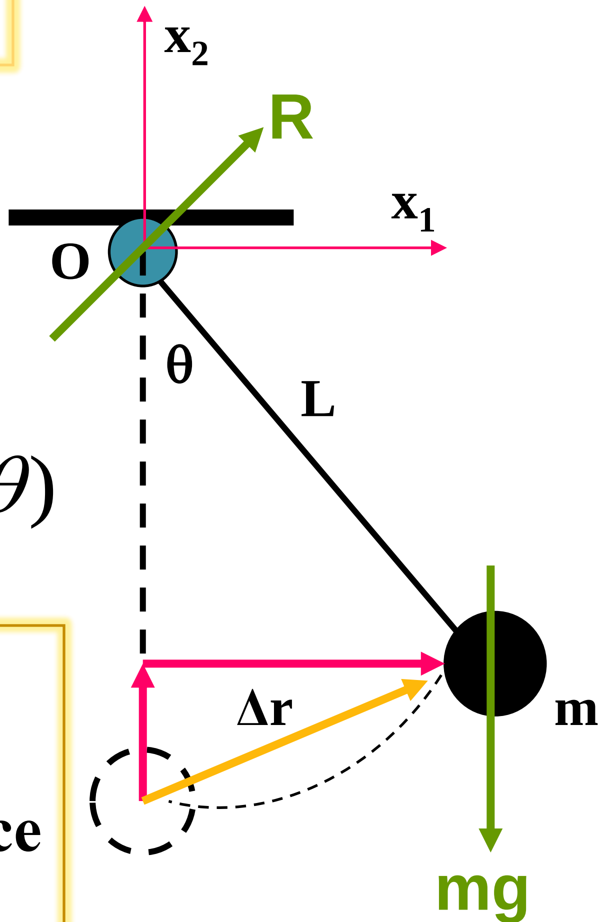
Method (I): Refer to free-body and consider the path of the particle.

$-mg\underline{e}_2$: working force acting on the particle which displaces by:

$$\Delta\underline{r} = L\sin\theta\,\underline{e}_1 + L(1-\cos\theta)\,\underline{e}_2$$

$$U = -mg\underline{e}_2 \cdot \Delta\underline{r} = -mgL(1-\cos\theta)$$



Method (II): consider the pendulum assembly as a Rigid Body that pivots about "O". Then, the gravitational force has a moment about "O" as:

$$\underline{M}^O = -mgL\sin\theta\,\underline{e}_3 \qquad ,\omega = \dot{\theta}\,(ccw)$$

$$\dot{U} = \underline{\omega}\cdot\underline{M}^O = -mgL\dot{\theta}\sin\theta$$

(integrating gives the same result)

متشکرم

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