بسم الله الرحمن الرحیم

By: Professor Ali Meghdari

NON-NEWTONIAN REFERRENCE FRAME

Purpose:

- To Study Mechanics in ***Non-Newtonian Reference Frame*** **(NNRF).**

Topics:

- **Kinetic Principles of a *Particle* in a *NNRF*.**
- **Kinetic Principles of a *System of Particles* & *Rigid Bodies* in NNRF.**

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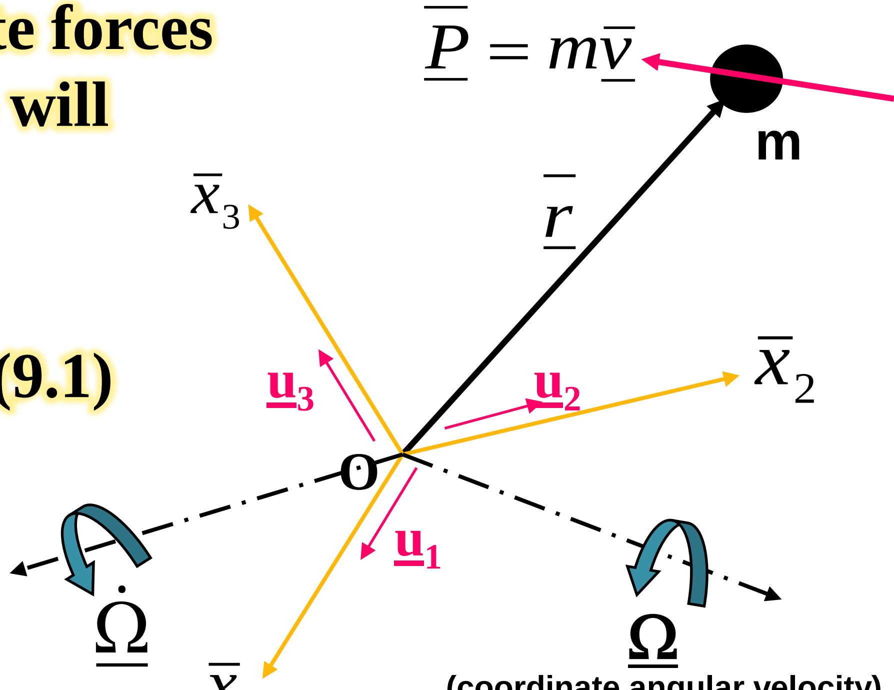
Kinetic Principles of a *Particle* in a NNRF

Remark: **In Kinetics analysis of a system, the admissibilityof certain forces on the free body diagram depends on the reference frame in which one experiences. In this chapterwe shall study other *admissible forces* in addition to *Newtonian Forces*, as well as their corresponding line of action.**

P-Principle: **When coordinate forces are admitted, the P-Principle will take the following form:**

$$\underline{\bar{f}} = \frac{d}{dt}(\underline{\bar{P}}) = \dot{\underline{\bar{P}}} = m\underline{\bar{a}} \qquad \textbf{(9.1)}$$



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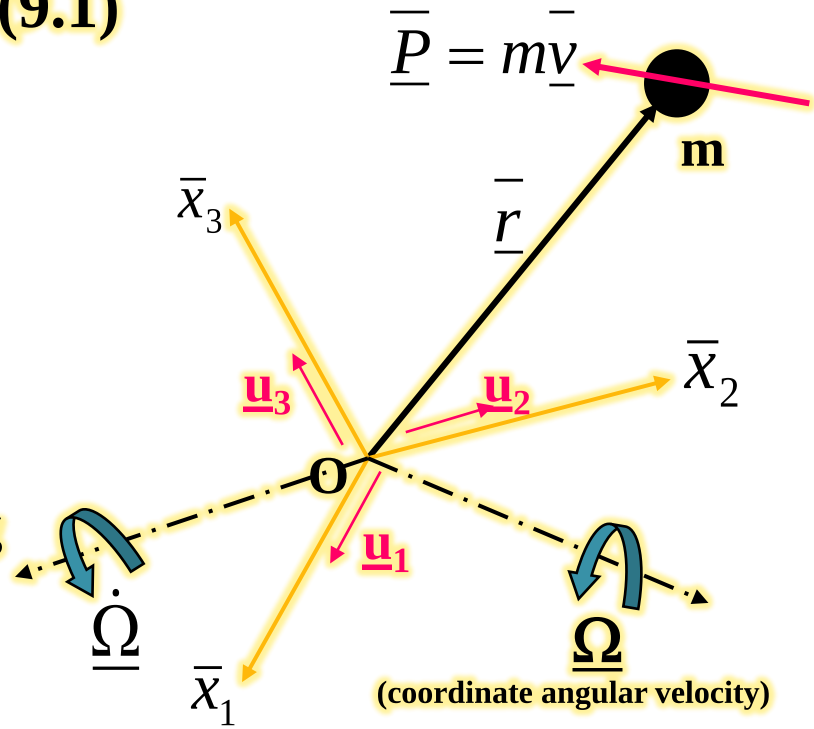
$$\bar{f} = \frac{d}{dt}(\bar{P}) = \dot{\bar{P}} = m\bar{a} \qquad (9.1)$$



Where:

- $\bar{a}$ **: Observed acc. In NNRF**
- $\bar{f}$ **: Admissible force field including both *Newtonian Forces* f and *Non-Newtonian Forces.***

From Chapter 5; we have the *absolute acceleration* of the particle computed as:

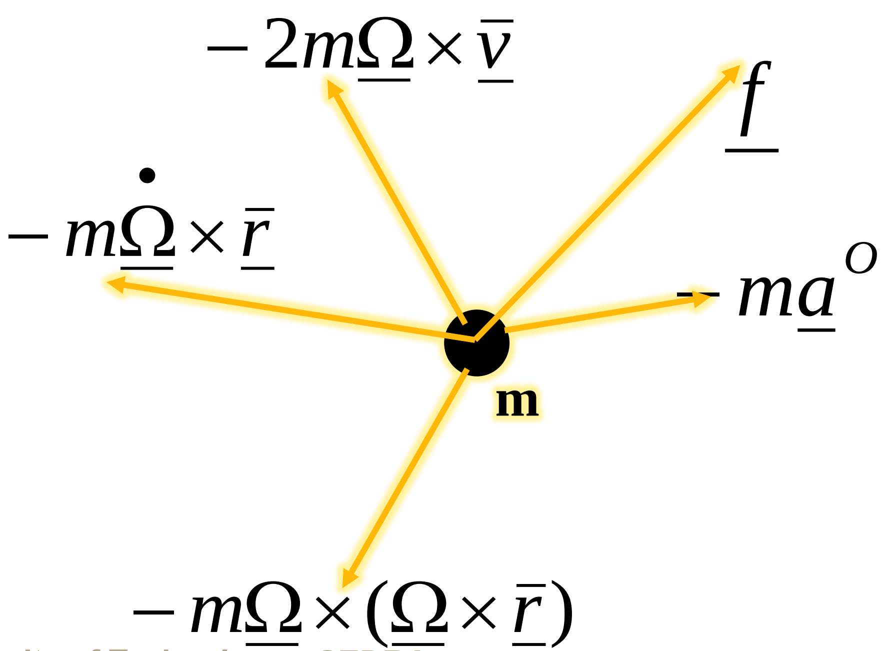
$$a = a_O + \bar{a} + \dot{\Omega} \times \bar{r} + 2\Omega \times \bar{v} + \Omega \times (\Omega \times \bar{r})$$

Applying Newton’s Equation results:

$$\underline{f} = m\underline{a} = m\underline{a}^O + \underbrace{m\underline{\bar{a}}}_{\underline{\bar{f}}} + m\underline{\dot{\Omega}}\times\underline{\bar{r}} + 2m\underline{\Omega}\times\underline{\bar{v}} + m\underline{\Omega}\times(\underline{\Omega}\times\underline{\bar{r}}) \qquad \textbf{(9.2)}$$

Therefore:

$$\underline{\bar{f}} = m\underline{\bar{a}} = \underline{f} - m\underline{a}^O - m\underline{\dot{\Omega}}\times\underline{\bar{r}} - 2m\underline{\Omega}\times\underline{\bar{v}} - m\underline{\Omega}\times(\underline{\Omega}\times\underline{\bar{r}}) \qquad \textbf{(9.3)}$$



Admissible Forces **all acting on the particle of mass "m" in the *NNRF* are:**

$\underline{f}$	**Newtonian Forces**
$-m\underline{a}^{O}$	**The Coordinate Inertia**
$-m\dot{\underline{\Omega}}\times\underline{\bar{r}}$	**The Euler's Force**
$-2m\underline{\Omega}\times\underline{\bar{v}}$	**The Coriolis Force**
$-m\underline{\Omega}\times(\underline{\Omega}\times\underline{\bar{r}})$	**The Centrifugal Inertia**

Notation Remarks:

$\overline{\bullet}$: Time rate of change of the relative quantity in the absolute sense. For vector quantities, the rotation of the reference frame is to be considered. (**Ex:** $\underline{\overline{\dot{H}}}$)

$\underline{\bullet}$: "Observed" time variation of the relative quantity in the reference frame. Hence, the time derivative is independent of the rotation of the reference frame.

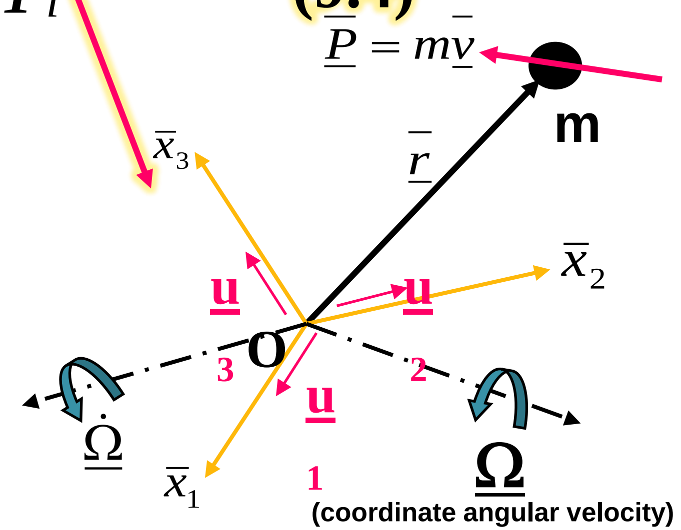
Ex: $\underline{\dot{\overline{H}}}$: Observed time variation of $\overline{H}$ in the NNRF (independent of the rotation of the reference frame)

H-Principle: Remember that $\{\underline{u}_i\}$ is fixed to the NNRF.

***Theorem-29*:** With the *Non-Newtonian* forces acting on the particle, the *observed* time rate of change in *Moment-of-Momentum* is equal to the admissible *Moment-of-Force* about the same moment center fixed in the NNRF.

$$\underline{\overline{M}}^{O} = \dot{\overline{H}}_i^{O} \, \underline{u}_i \quad \text{or} \quad \overline{M}_i^{O} = \dot{\overline{H}}_i^{O} \qquad (9.4)$$

Where: "O" is A fixed point in the NNRF.



Observed time variation of $\overline{H}$ in the NNRF.

(independent of the rotation of the reference frame)

Proof:

The observed H about "O" is:

$$\underline{\overline{H}}^{O} = \overline{H}^{O}_{i}\, \underline{u}_{i} = \underline{\overline{r}} \times \underline{\overline{P}} = \underline{\overline{r}} \times m\underline{\overline{v}} \qquad \text{(a)}$$

Its time rate of change as a variable vector in a rotating coordinate is:

$$\frac{d}{dt}\underline{\overline{H}}^{O} = \underline{\dot{\overline{H}}}^{O} = \dot{\overline{H}}^{O}_{i}\, \underline{u}_{i} + \underline{\Omega} \times \underline{\overline{H}}^{O} \qquad \text{(b)}$$

Or by time derivative of equation (a) as:

$$\frac{d}{dt}\underline{\overline{H}}^{O} = \underline{\dot{\overline{H}}}^{O} = \underline{\overline{r}} \times m\underline{\dot{\overline{v}}} + \underline{\dot{\overline{r}}} \times m\underline{\overline{v}} = \qquad \text{(c)}$$

$$= \underline{\overline{r}} \times [m(\underline{\overline{a}} + \underline{\Omega} \times \underline{\overline{v}})] + [\underline{\overline{v}} + (\underline{\Omega} \times \underline{\overline{r}})] \times m\underline{\overline{v}}$$

Since: $(\underline{A}\times\underline{B}) = -(\underline{B}\times\underline{A})$

$$\underline{A}\times(\underline{B}\times\underline{C}) + \underline{B}\times(\underline{C}\times\underline{A}) + \underline{C}\times(\underline{A}\times\underline{B}) = 0$$

$$\frac{d}{dt}\overline{\underline{H}}^{O} = \overline{\underline{\dot{H}}}^{O} = \underline{\bar{r}}\times \underbrace{m\underline{\bar{a}}}_{\overline{f}} + \underbrace{m[\underline{\bar{v}}\times(\underline{\bar{r}}\times\underline{\Omega}) + \underline{\bar{r}}\times(\underline{\Omega}\times\underline{\bar{v}})]}_{-\underline{\Omega}\times(\underline{\bar{v}}\times\underline{\bar{r}}) = \underline{\Omega}\times(\underline{\bar{r}}\times\underline{\bar{v}})}$$

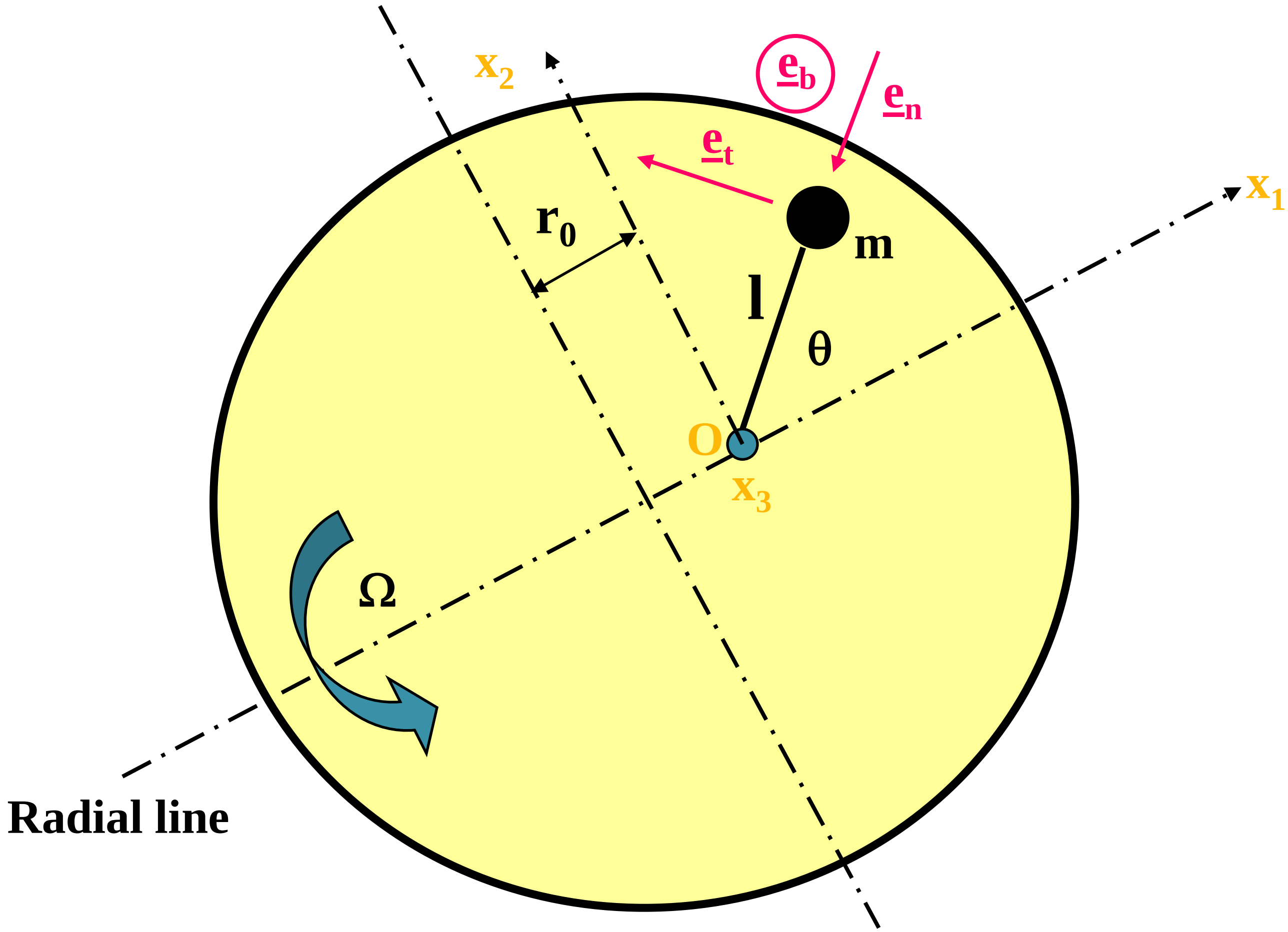
$$\underbrace{\underline{\bar{r}}\times\underline{\overline{f}}}_{\overline{\underline{M}}^{O}}$$

$$\overline{\underline{\dot{H}}}^{O} = \overline{\underline{M}}^{O} + \underline{\Omega}\times\underbrace{(\underline{\bar{r}}\times m\underline{\bar{v}})}_{\overline{\underline{H}}^{O}} = \overline{\underline{M}}^{O} + \underline{\Omega}\times\overline{\underline{H}}^{O} \qquad \text{(d)}$$

Comparing Equations (b) and (d) results:

$$\boxed{\overline{\underline{M}}^{O} = \overline{\dot{H}}^{O}_{i}\,\underline{u}_{i}}$$

Example: In a saucer-shaped spaceship in isotropic space, a crew member hangs a pendulum at a distance r_0 from the spinning axis. Determine the period of oscillation in the equatorial plane. The ship rotates at a uniform speed of Ω.

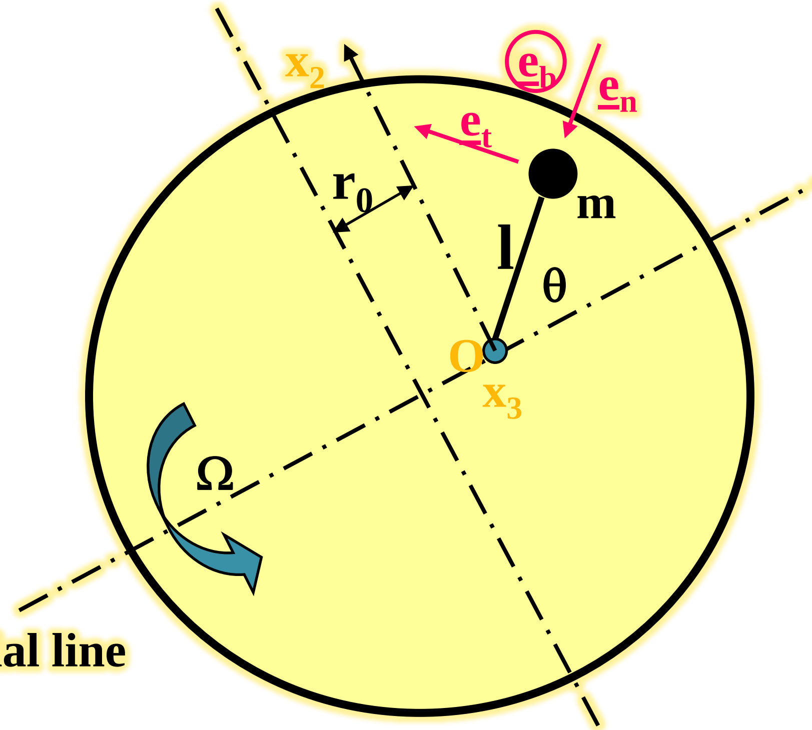


Solution:

1. Choose NNRF on the spaceship with Ref. Point at "O".

$$\underline{\Omega} = \Omega \underline{e}_3 = \Omega \underline{e}_b \,, \quad \dot{\underline{\Omega}} = 0$$

$$\underline{a}_O = -r_0 \Omega^2 \underline{e}_1$$

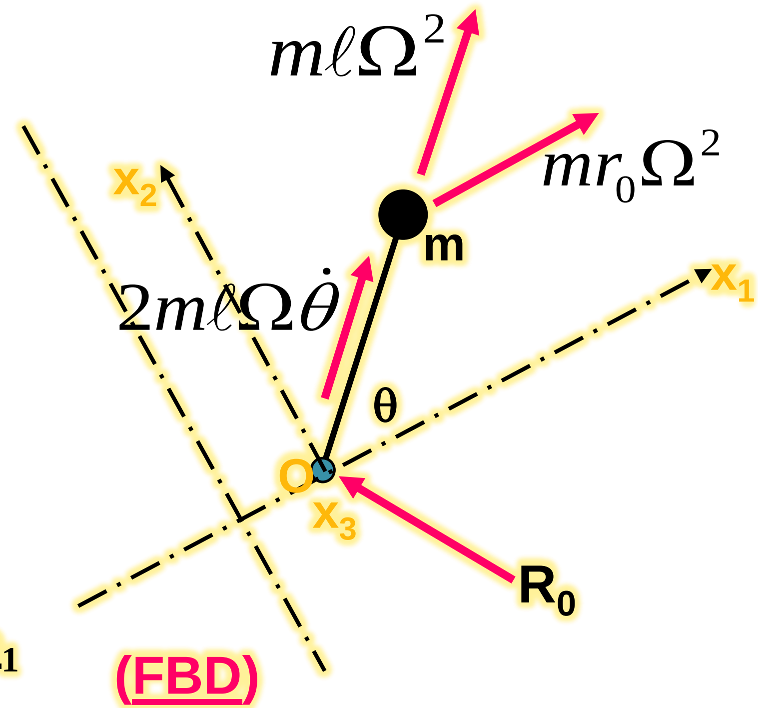


2. Relative (observed) Motion:

$$\underline{\bar{r}} = -\ell \underline{e}_n \,, \quad \underline{\bar{v}} = \ell \dot{\theta} \underline{e}_t$$

$$\underline{\bar{a}} = \ell \ddot{\theta} \underline{e}_t + \ell \dot{\theta}^2 \underline{e}_n$$

3. Free Body Diagram: Showing Admissible Forces;



$\underline{f}$ **: Reaction Force at "O" = $\underline{R}_0$**

$-m\underline{a}^O$ **: Parallel to Radial Line= $mr_0\Omega^2\underline{e}_1$**

$$-m\dot{\underline{\Omega}}\times\underline{\bar{r}}=0$$

$$-2m\underline{\Omega}\times\underline{\bar{v}}=-2m(\Omega\underline{e}_b)\times(\ell\dot{\theta}\underline{e}_t)=-2m\ell\Omega\dot{\theta}\underline{e}_n$$

$$-m\underline{\Omega}\times(\underline{\Omega}\times\underline{\bar{r}})=-m(\Omega\underline{e}_b)\times[\Omega\underline{e}_b\times(-\ell\underline{e}_n)]=-m\ell\Omega^2\underline{e}_n$$

4. **Kinetics Analysis:**

$$\underline{\overline{M}}^O = \dot{\overline{H}}_i^O \underline{u}_i \quad \text{or} \quad \overline{M}_i^O = \dot{\overline{H}}_i^O$$

$$\underline{\overline{H}}^O = \underline{\overline{r}} \times m\underline{\overline{v}} = (-\ell \underline{e}_n) \times (m\ell\dot{\theta}\underline{e}_t)$$

$$= m\ell^2\dot{\theta}\underline{e}_b = m\ell^2\dot{\theta}\underline{e}_3$$

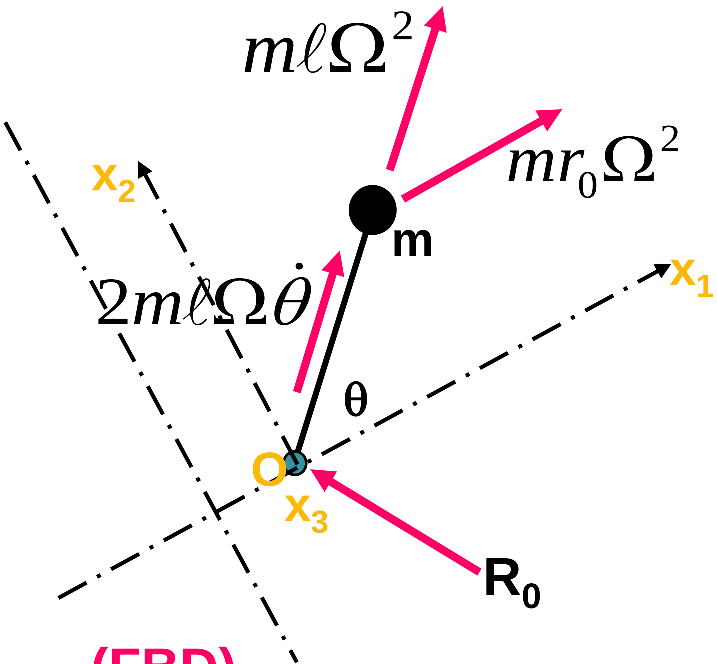
Since: $\overline{M}_3^O = \dot{\overline{H}}_3^O$

$$\begin{cases} \overline{H}_3^O = m\ell^2\dot{\theta} \Rightarrow \dot{\overline{H}}_3^O = m\ell^2\ddot{\theta} \\ \overline{M}_3^O = -(mr_0\Omega^2)(\ell\sin\theta) \end{cases}$$

$$\begin{cases} \ddot{\theta} + \left(\dfrac{r_0\Omega^2}{\ell}\right)\sin\theta = 0 \\ \omega_n^2 = \dfrac{r_0\Omega^2}{\ell} \Rightarrow \omega_n = \Omega\sqrt{\dfrac{r_0}{\ell}} \Rightarrow \tau = \dfrac{2\pi}{\omega_n} = \dfrac{2\pi}{\Omega}\sqrt{\dfrac{\ell}{r_0}} \end{cases}$$

5. Alternate Method: By *Momentum Principle*;

$$\underline{\overline{f}} = m\underline{\overline{a}}$$



(FBD)

Coefficients of $\underline{e}_t$:

$$-mr_0\Omega^2 \sin\theta = m\ell^2\ddot{\theta}$$

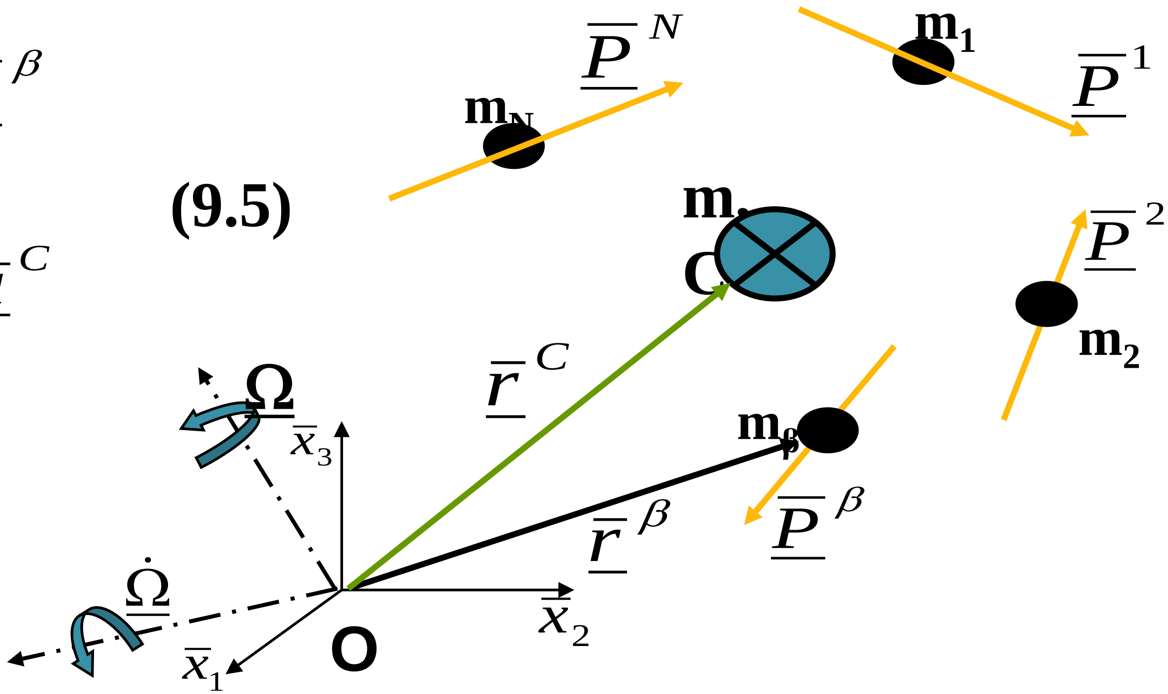
$$\begin{cases} \ddot{\theta} + \left(\dfrac{r_0\Omega^2}{\ell}\right)\sin\theta = 0 \\ \omega_n^2 = \dfrac{r_0\Omega^2}{\ell} \Rightarrow \omega_n = \Omega\sqrt{\dfrac{r_0}{\ell}} \Rightarrow \tau = \dfrac{2\pi}{\omega_n} = \dfrac{2\pi}{\Omega}\sqrt{\dfrac{\ell}{r_0}} \end{cases}$$

Kinetic Principles in a NNRF for a System of *Particles & Rigid Bodies*

P-Principle for a System of Particles: similar to P-Principle in NRF, the equation relates the Global Admissible Forces to the change in Kinetic State of the *mass center*.

$$\underline{\overline{P}} = m\underline{\overline{v}}^{C} = \sum_{\beta=1}^{N} \underline{\overline{P}}^{\beta}$$

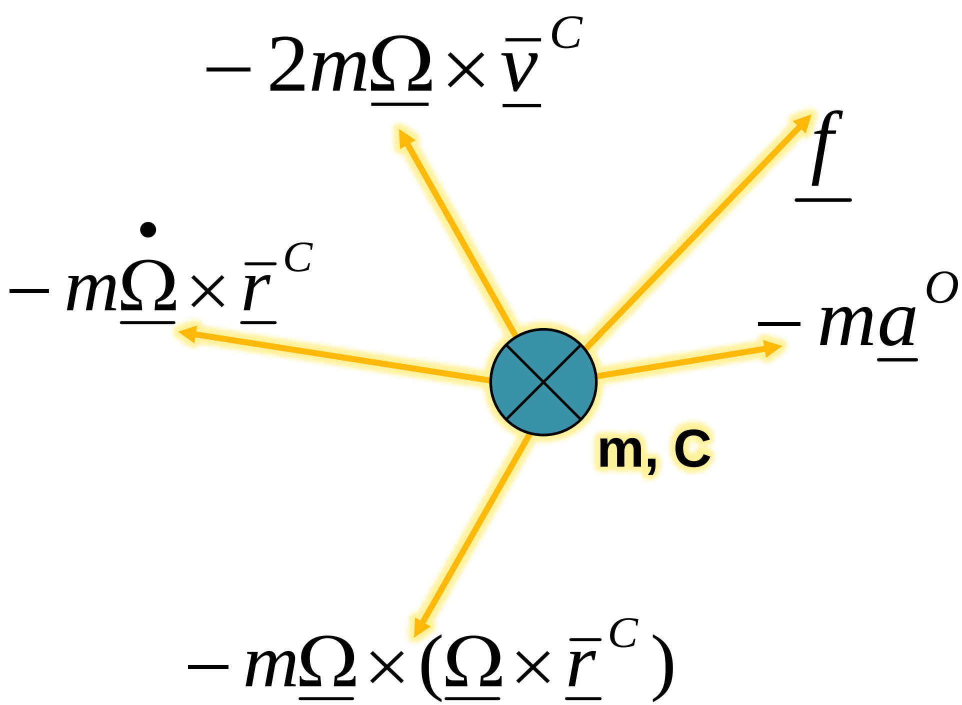
$$\underline{\overline{f}} = \frac{d}{dt}(\underline{\overline{P}}) = m\underline{\overline{a}}^{C} \qquad (9.5)$$



The *Admissible Forces* are:

$$\overline{\underline{f}} = \frac{d}{dt}(\overline{\underline{P}}) = m\overline{\underline{a}}^{C}$$

$$\overline{\underline{f}} = \underline{f} - m\underline{a}^{O} - m\dot{\underline{\Omega}} \times \overline{\underline{r}}^{C} - 2m\underline{\Omega} \times \overline{\underline{v}}^{C} - m\underline{\Omega} \times (\underline{\Omega} \times \overline{\underline{r}}^{C}) \qquad \textbf{(9.6)}$$



Note: All Global Coordinate Forces act at the mass center of the material system.

H-Principle for a System of Particles: **with respect to a moment center fixed to the NNRF, the observed time rate of change in moment-of-momentum is equal to the *totality of* (effect of) all admissible moments.**

$$\underline{\overline{H}}^{O} = \sum_{\beta=1}^{N} \underline{\overline{r}}^{\beta} \times m_{\beta} \underline{\overline{v}}^{\beta}$$ ⟺ **{Global Moment-of-Momentum}**

$$\underline{\overline{M}}^{O} = \underline{\dot{\overline{H}}}_{i} \underline{u}_{i} \quad or \quad \underline{\overline{M}}_{i}^{O} = \underline{\dot{\overline{H}}}_{i}^{O} \qquad \textbf{(9.7)}$$

{Global Moment-of-Forces}

Admissible Forces on the Free Body Diagram are:

1. **Newtonian Forces:** $\underline{f}$

2. **Non-Newtonian Forces:** $\underline{\bar{f}}_{\beta} = m_{\beta}\underline{\bar{a}}^{\beta}$

   a) Individual Particle Expression:

$$\{m_{\beta}\underline{a}^{O}\},$$

$$\{-m_{\beta}\underline{\dot{\Omega}}\times\underline{\bar{r}}^{\beta}\},$$

$$\{-m_{\beta}\underline{\Omega}\times(\underline{\Omega}\times\underline{\bar{r}}^{\beta})\},$$

$$\{-2m_{\beta}\underline{\Omega}\times\underline{\bar{v}}^{\beta}\}$$

Admissible Forces on the Free Body Diagram are:

b) Global Expression (all particles)

$$\overline{\underline{f}} = \sum_{\beta=1}^{N} \overline{\underline{f}}_{\beta} = \sum_{\beta=1}^{N} m_{\beta} \overline{\underline{a}}^{\beta} =$$

$$= \sum_{\beta=1}^{N} \underline{f}_{\beta} - \sum_{\beta=1}^{N} m_{\beta} \underline{a}^{O} - \dot{\underline{\Omega}} \times \sum_{\beta=1}^{N} m_{\beta} \overline{\underline{r}}^{\beta} - 2\underline{\Omega} \times \sum_{\beta=1}^{N} m_{\beta} \overline{\underline{v}}^{\beta} - \underline{\Omega} \times (\underline{\Omega} \times \sum_{\beta=1}^{N} m_{\beta} \overline{\underline{r}}^{\beta})$$

$$\overline{\underline{f}} = \underline{f} - m\underline{a}^{O} - m\dot{\underline{\Omega}} \times \overline{\underline{r}}^{C} - 2m\underline{\Omega} \times \overline{\underline{v}}^{C} - m\underline{\Omega} \times (\underline{\Omega} \times \overline{\underline{r}}^{C})$$

(9.6)

And from H-Principle:

{ith-component}

$$\overline{M}_{i}^{O} = [\sum_{\beta=1}^{N} \overline{\underline{M}}_{\beta}^{O}]_{i} = \dot{\overline{H}}_{i}^{O}$$

(9.8)

متشکرم