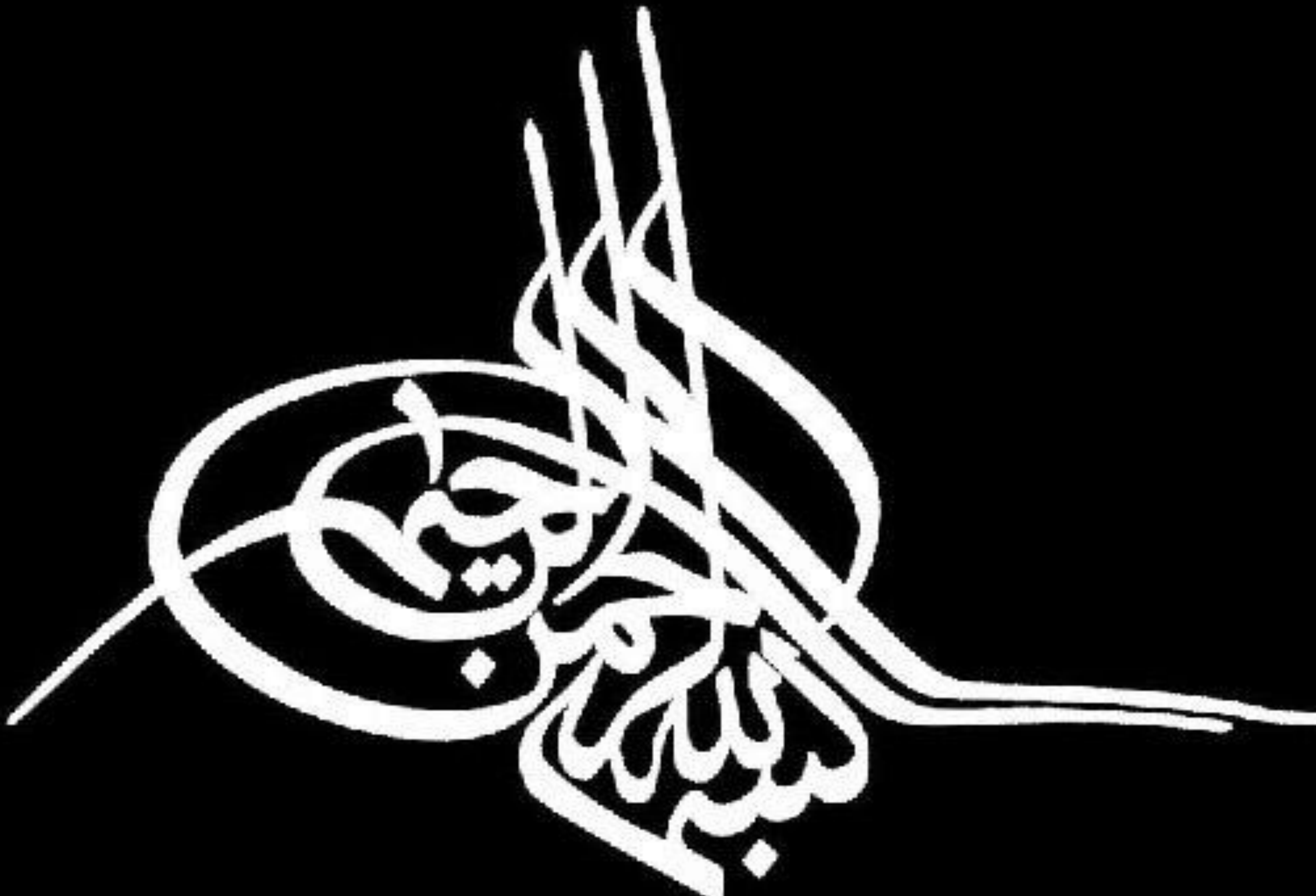
بسم الله الرحمن الرحيم

RIGID BODY DYNAMICS

Purpose:

- **To Study Kinetic States and Principles of Rigid Bodies.**

Topics:

- **Kinetic States of a Rigid Body.**
- **Kinetic Principles of a Rigid Body.**
- **Rigid Body Rotation about an Invariant Axis.**

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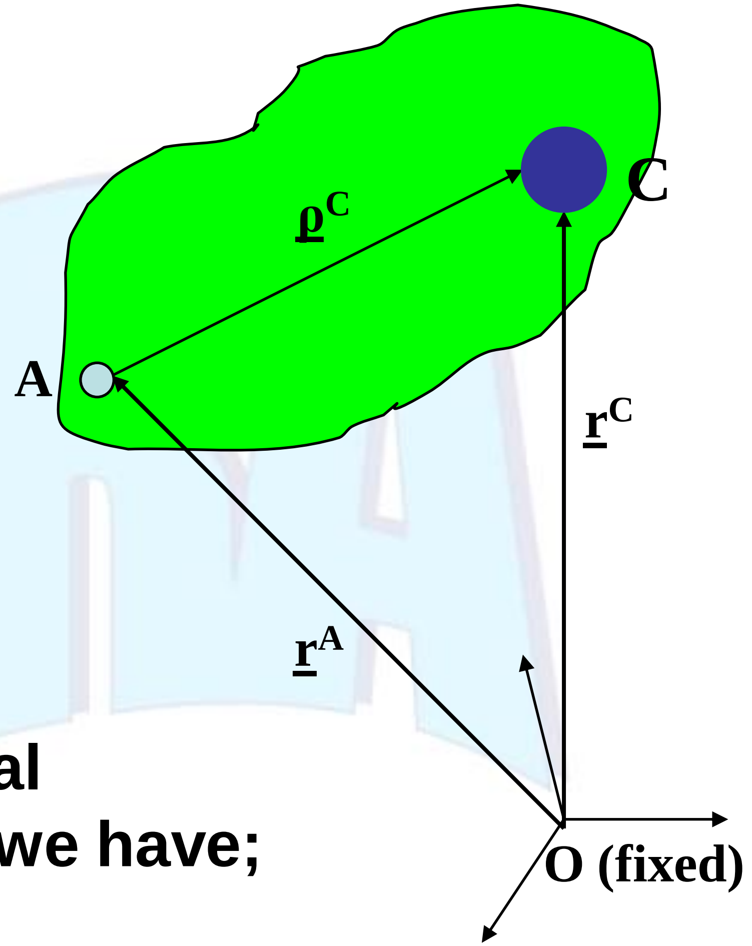
Kinetic Principles of a Rigid Body: Consider a rigid body as shown, where point "A" is a body point, then:



Momentum Principle (P-Principle) for the Rigid Body is:

$$\underline{f} = m\underline{a}^{C} = \underline{\dot{P}} \qquad \textbf{(8.7)}$$

Momentum of Momentum Principle (H-Principle) for the Rigid Body is:

In Chapter-6 we showed that for a system of particles, with the general moment center "A", and $\underline{P} = m\underline{v}^{C}$, we have;

$$\underline{M}^{A} = \underline{\dot{H}}^{A} + (\underline{v}^{A} \times \underline{P}) \qquad \textbf{(6.23)}$$

However, if

$$\left.\begin{cases} \text{"}A\text{"}: \quad is-a-fixed-point, \quad or \\ A \equiv C, \quad or \\ \underline{v}^A = 0 \quad or \quad \underline{v}^A \,\|\, \underline{v}^C, and \quad \underline{\rho}^C \,\|\, \underline{a}^A \end{cases}\right\} \Rightarrow \underline{M}^A = \underline{\dot{H}}^A = \frac{d}{dt}(\underline{\underline{I}}^A \cdot \underline{\omega})$$

Then:

$$\underline{M}^A = \underline{\dot{H}}^A = \frac{d}{dt}(\underline{\underline{I}}^A \cdot \underline{\omega}), \quad or \quad M_i^A = \frac{d}{dt}(I_{ij}^A \omega_j) \qquad \textbf{(8.9)}$$

Equation (8.9) is referred to as the *Euler's Equation*.

***Definition*: A moment center satisfying the conditions stated above is called an *Eulerian Moment Center*.**

Generalized form of Euler's Equation:

Moment of Momentum Principle (*$\underline{H}$-Principle*) in terms of a rotating coordinate {x_i} with angular velocity $\underline{\Omega}$ may be written as:

$$\underline{M}^A = \dot{H}_i^A \underline{u}_i + \underline{\Omega} \times \underline{H}^A \qquad where \quad H_i^A = I_{ij}^A \omega_j \qquad \textbf{(8.11)}$$

Let {x_i} to be the body coordinate (fixed to the body), then: $\underline{\Omega} \longrightarrow \underline{\omega}$, and {I_{ij}^A } will form a *constant set*. Therefore:

$$\underline{M}^A = \frac{d}{dt}(I_{ij}^A \omega_j)\underline{u}_i + \underline{\omega} \times (\underline{\underline{I}}^A \cdot \underline{\omega}) = I_{ij}^A \alpha_j \underline{u}_i + \underline{\omega} \times (\underline{\underline{I}}^A \cdot \underline{\omega}) =$$

$$\underline{M}^{A} = \underline{\underline{I}}^{A} \cdot \underline{\alpha} + \underline{\omega} \times (\underline{\underline{I}}^{A} \cdot \underline{\omega}) \qquad or \qquad M_i^A = I_{ij}^A \alpha_j + \gamma_{ijk} \omega_j I_{ks}^A \omega_s$$

(8.12)

{*Very Important*, *Gen. Form of Euler's Equation*, *Actually Coordinate Independent* }

In terms of *Principal Coordinates at the Mass Center, or a Fixed Point* "A" in a rigid body that is in *a state of pure rotation*, Equation (8.12) takes on the form:

$$I_{11}^A \to I_1^A, I_{22}^A \to I_2^A, I_{33}^A \to I_3^A, I_{12}^A = I_{23}^A = I_{31}^A = 0$$ **, and**

$$M_1^A = I_1^A \alpha_1 + \gamma_{12k} \omega_2 I_{ks}^A \omega_s + \gamma_{13k} \omega_3 I_{ks}^A \omega_s = I_1^A \alpha_1 + I_3^A \omega_2 \omega_s - I_2^A \omega_2 \omega_3$$

A More Familiar Form of *Euler's Equations of Motion*:

$$M_1^A = I_1^A \alpha_1 - (I_2^A - I_3^A)\omega_2\omega_3$$
$$M_2^A = I_2^A \alpha_2 - (I_3^A - I_1^A)\omega_1\omega_3 \qquad \textbf{(8.13)}$$
$$M_3^A = I_3^A \alpha_3 - (I_1^A - I_2^A)\omega_1\omega_2$$

Note: *Free Rotation* of a rigid body is generally due to the absence of an external resultant moment about its mass center. Many dynamic problems are of this nature, and the only force applied to them is from gravity. In these cases, *Euler's Equations* about the *principal axes at the mass center* will take the following form:

$$I_1^C \alpha_1 = (I_2^C - I_3^C)\omega_2\omega_3$$
$$I_2^C \alpha_2 = (I_3^C - I_1^C)\omega_1\omega_3 \qquad \textbf{(8.14)}$$
$$I_3^C \alpha_3 = (I_1^C - I_2^C)\omega_1\omega_2$$

In other words, in order for such a situation to exist, moment of momentum about the mass center must be constant ($\underline{H}^C$ = constant), similarly the system must also be conservative (*meaning that the sum of potential and kinetic energy of the system must always be constant*), and the total kinetic energy of rotation must also be constant.

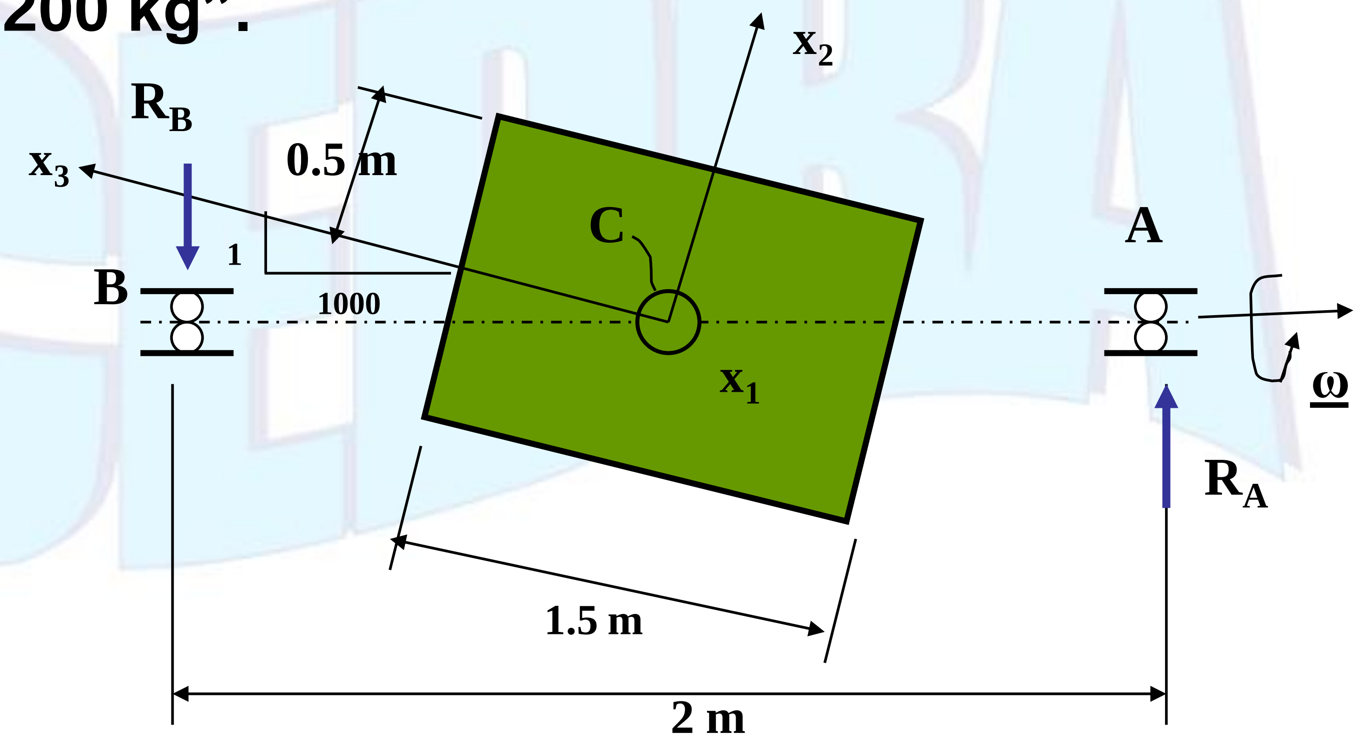
$$\begin{cases} \underline{H}^C = \underline{\textit{Constant}} \\ \text{Total K.E.})_{\text{Rot.}} = \underline{\textit{Constant}} \\ \text{K.E.})_{\text{Sys.}} + \text{P.E.})_{\text{Sys.}} = \underline{\textit{Constant}} = \text{(conservative system)} \end{cases}$$

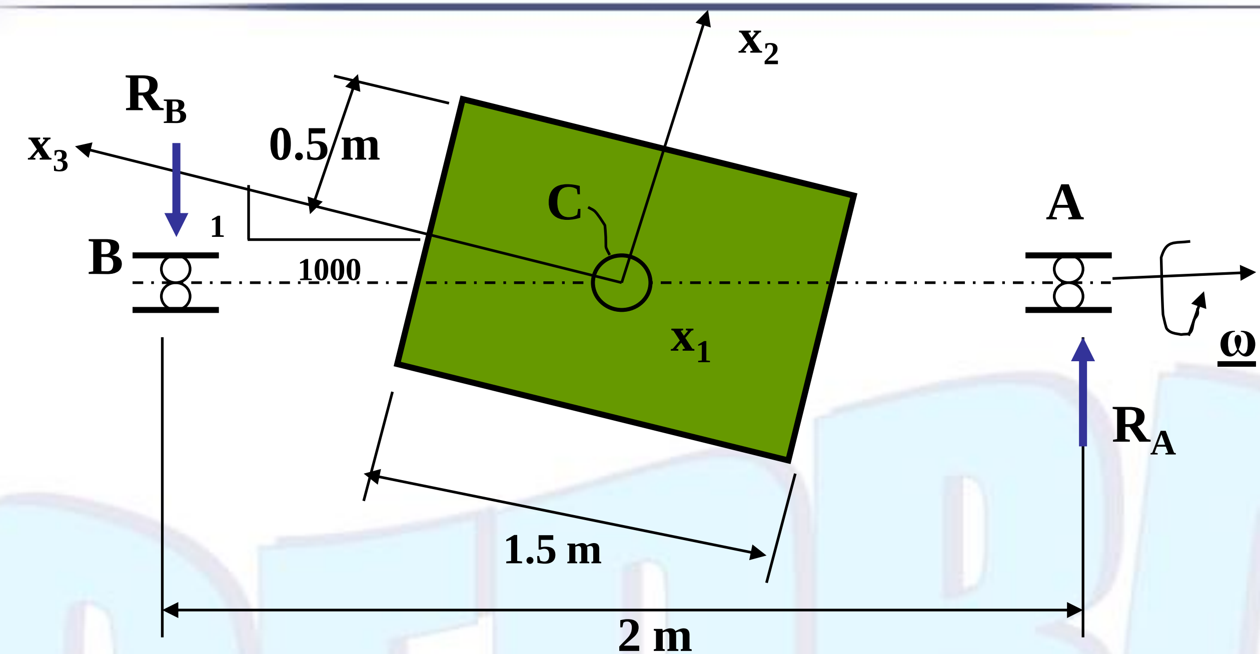
Equations (8.7) and (8.9) together (Newton and Euler's Equations), form the *General Equations of Motion* for a Rigid Body.

Example:

A turbine rotor assembly is dynamically equivalent to a solid rotating cylinder. The mounting is inadvertently misaligning by “0.1%” from the axis of rotating symmetry. Determine the bearing load due to the unbalance at a uniform speed of “1000 rad/sec”. The rotor assembly has a mass of “200 kg”.

Solution:





In terms of the principal body coordinates {x_i}, the principal moments of inertias are:

$$I_1^C = I_2^C = m(\frac{1}{4}R^2 + \frac{1}{12}L^2) = 50 \quad kg.m^2, and \quad I_3^C = \frac{1}{2}mR^2 = 25 \quad kg.m^2$$

The *angular velocity vector* "$\underline{\omega}$" is:

$$\underline{\omega} \cong \frac{\omega}{1000}\underline{u}_2 - \omega\underline{u}_3 = \underline{u}_2 - 1000\underline{u}_3 \quad rad/\sec, and \quad \underline{\alpha} = 0$$ **(ω is uniform).**

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Now, applying *Eulerian Equation* we have:

$$M_1^C = I_1^C \alpha_1 - (I_2^C - I_3^C)\omega_2\omega_3 = 0 - (50-25)(1)(-1000) = 25000 \qquad N.m$$

$$M_2^C = M_3^C = 0$$

This moment is provided by bearing reactions. Since the *moment arm* = 2 m, then:

$$R_A = R_B = 12500 \qquad N, \qquad \perp$$ ***to the spin axis***

Therefore, the unbalanced force is cycling at a rate of 1000 rad/sec = 9550 RPM, where the shaking force is quite damaging.

Rigid Body Rotation about an Invariant Axis:

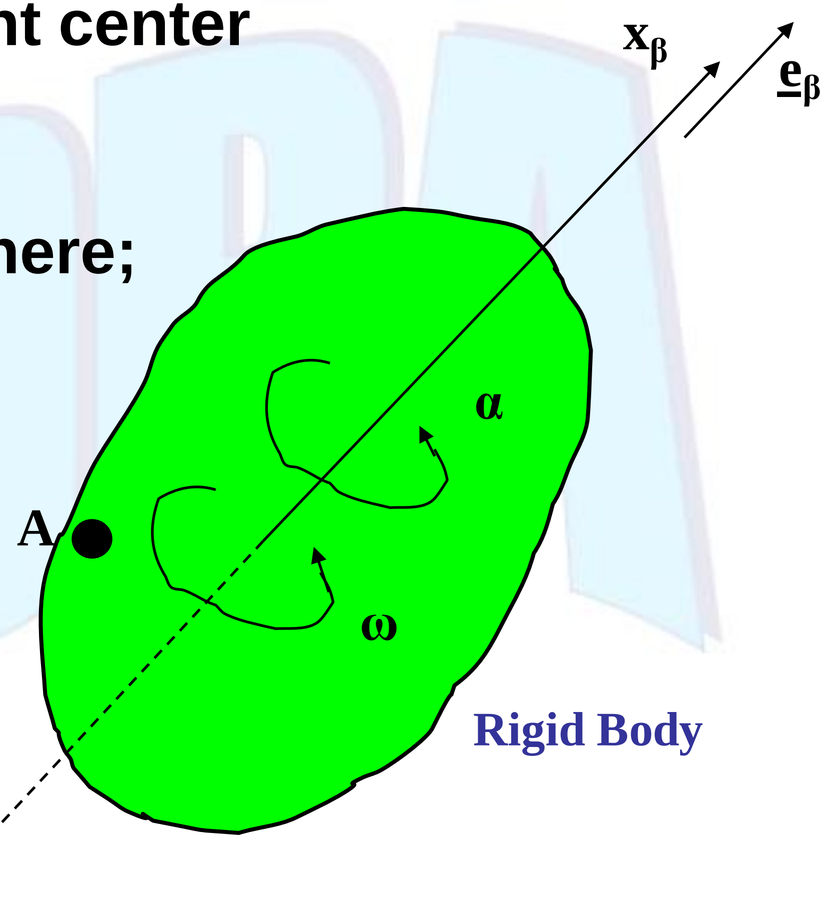
Consider a rigid body as shown:

Let: "x_β" be the invariant axis of rotation, and "A" be an *Eulerian* moment center for the rigid body, therefore:

$\underline{\omega} = \omega \underline{e}_{\beta}$ *and* $\underline{\alpha} = \alpha \underline{e}_{\beta}$, **where;**

$\underline{e}_\beta$ is *constant*, and therefore $\underline{\omega}$ and $\underline{\alpha}$ have *invariant directions*.

Now, let us choose a body coordinate $\{x_i\}$ such that $\underline{u}_\beta = \underline{e}_\beta$, thus:



$$\begin{cases} \underline{\omega} = \omega_i \underline{u}_i = \omega \underline{u}_\beta = \omega \delta_{i\beta} \underline{u}_i \Rightarrow \omega_i = \omega \delta_{i\beta} \\ \underline{\alpha} = \alpha_i \underline{u}_i = \alpha \underline{u}_\beta = \alpha \delta_{i\beta} \underline{u}_i \Rightarrow \alpha_i = \alpha \delta_{i\beta} \end{cases}$$

From the General Euler's Equation $M_i^A = I_{ij}^A \alpha_j + \gamma_{ijk} \omega_j I_{ks}^A \omega_s$ **, we have:**

$$M_i^A = I_{ij}^A \overbrace{\delta_{j\beta}\alpha}^{\alpha_j} + \gamma_{ijk} \overbrace{\delta_{j\beta}\omega}^{\omega_j} I_{ks}^A \overbrace{\delta_{s\beta}\omega}^{\omega_s}$$

$$M_i^A = I_{i\beta}^A \alpha + \gamma_{i\beta k} I_{k\beta}^A \omega^2 \qquad \textbf{(8.15)}$$

for β = 1, we have:

$$\begin{cases} M_1^A = I_{11}^A \alpha + 0 \\ M_2^A = I_{21}^A \alpha - I_{31}^A \omega^2 \\ M_3^A = I_{31}^A \alpha + I_{21}^A \omega^2 \end{cases} \qquad \textbf{(8.16)}$$

for β = 2, we have:

$$\begin{cases} M_1^A = I_{12}^A \alpha + I_{32}^A \omega^2 \\ M_2^A = I_{22}^A \alpha \\ M_3^A = I_{32}^A \alpha - I_{21}^A \omega^2 \end{cases} \qquad (8.17)$$

Note that: in planar motion if "x_β" is *not* the principal axes, then products of inertia exists, and if α = 0, that does not mean that $\underline{M} = 0$.

***Theorem-26*: For a rigid body in plane motion, the set of kinetic equations are:**

$$\begin{aligned} \underline{F} &= m\underline{a}^C \\ M^O &= I^O \alpha \end{aligned} \qquad (8.18)$$

$$\underline{F} = m\underline{a}^{C}$$
$$M^{O} = I^{O}\alpha \qquad \textbf{(8.18)}$$

where; M_O: Moment about an axis $\perp$ to the plane of motion at "O".

I_O : Moment of Inertia about an axis $\perp$ to the plane of motion at "O".

O : An *Eulerian* Moment Center.

***Theorem-27*: Equation (8.15) implies that *momentless rotation (free-rotation)* about an axis of fixed orientation is possible provided that the axis is a principal axis of inertia of the rigid body at any point on the axis of rotation.**

Theorem-28: *Free Rotation* of a rigid body about an invariant axis is *stable* provided that the axis of rotation is a central principal axis corresponding to either the maximum or the minimum principal values.

***Proof*:**

Consider a rigid body, and let it spin about its central principal axis "x_1".

Originally let it have an angular velocity:

$$\underline{\omega}(0) = \{\omega_0, 0, 0\} = \omega_0 \underline{u}_1$$

Now, introduce the disturbance:

$$\Delta\omega = O(\varepsilon) \qquad , \Delta\omega << \omega_0$$ **, such that:**

$$\underline{\omega}(t) = \underline{\omega} + \Delta\underline{\omega} = (\omega_0 + \Delta\omega_1)\underline{u}_1 + \Delta\omega_2\underline{u}_2 + \Delta\omega_3\underline{u}_3$$ **, and**

$$\underline{\alpha}(t) = \Delta\dot{\omega}_1\underline{u}_1 + \Delta\dot{\omega}_2\underline{u}_2 + \Delta\dot{\omega}_3\underline{u}_3$$

Note that: prior and after disturbance, there is no external moment. From *Euler's Equation*, we have:

$$(a) \qquad M_1^C = 0 = I_1^C\Delta\dot{\omega}_1 - (I_2^C - I_3^C)\underbrace{\Delta\omega_2\Delta\omega_3}_{O(\varepsilon^2)\approx 0} \qquad \Rightarrow \Delta\omega_1 = const. << \omega_0 \Rightarrow$$

$$\omega_1 = \omega_0 + \underbrace{const.}_{\text{(so small)}} \cong \omega_0$$

$$(b) \qquad M_2^C = 0 = I_2^C\Delta\dot{\omega}_2 - (I_3^C - I_1^C)\omega_0\Delta\omega_3$$

$$(c) \qquad M_3^C = 0 = I_3^C\Delta\dot{\omega}_3 - (I_1^C - I_2^C)\omega_0\Delta\omega_2$$

Now we have;

$$(I_3^C)(\dot{b}) + (I_3^C - I_1^C)\omega_0(c) = 0 \quad \Rightarrow \quad \Delta\ddot{\omega}_2 + \frac{(I_1^C - I_3^C)(I_1^C - I_2^C)}{I_2^C I_3^C}\omega_0^2\Delta\omega_2 = 0$$

and similarly,

$$(I_2^C)(\dot{c}) + (I_1^C - I_2^C)\omega_0(b) = 0 \quad \Rightarrow \quad \Delta\ddot{\omega}_3 + \frac{(I_1^C - I_3^C)(I_1^C - I_2^C)}{I_2^C I_3^C}\omega_0^2\Delta\omega_3 = 0$$

Stability Condition **mandates that the coefficients of** $"\Delta\omega_2" or "\Delta\omega_3"$ **must be positive. Therefore;**

$$(I_1^C - I_3^C)(I_1^C - I_2^C) > 0, \quad where \quad \underline{\underline{I}}_\alpha^C = \{I_1^C, I_2^C, I_3^C\}$$

This condition holds if and only if the "I_1^C" is the *Maximum* or *Minimum* principal value (a mathematical idealization).

Note that: in reality due to internal damping there is only one stable axis of rotation, and that is the principal axis corresponding to the ***maximum principal value of inertia*****.**


متشکرم