بسم الله الرحمن الرحيم

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RIGID BODY DYNAMICS

Purpose:

- **To Study Kinetic States and Principles of Rigid Bodies.**

Topics:

- **Kinetic States of a Rigid Body.**
- **Kinetic Principles of a Rigid Body.**
- **Rigid Body Rotation about an Invariant Axis.**

***Kinetic States of a Rigid Body*:**

***Theorem-23*: The *Momentum* of a rigid body is equal to its mass times the velocity of the mass center.**

$$\underline{P} = m\underline{v}^{C} \qquad or \qquad P_i = mv_i^{C} \qquad \textbf{(8.1)}$$

***Theorem-24*: The *Central Moment of Momentum* for a rigid body is equal to its central inertia tensor times the angular velocity vector.**

$$\underline{H}^{C} = \underline{\underline{I}}^{C} \cdot \underline{\omega} \qquad or \qquad H_i^{C} = I_{ij}^{C}\omega_j \qquad i, j = 1,2,3 \qquad \textbf{(8.2)}$$

Equation (8.2) in expanded form is:

$$H_1^C = I_{11}^C\omega_1 + I_{12}^C\omega_2 + I_{13}^C\omega_3$$
$$H_2^C = I_{21}^C\omega_1 + I_{22}^C\omega_2 + I_{23}^C\omega_3 \qquad (8.3)$$
$$H_3^C = I_{31}^C\omega_1 + I_{32}^C\omega_2 + I_{33}^C\omega_3$$

Theorem-25: **The *Moment of Momentum* vector about a general body point "A" in the rigid body can be expressed as:**

$$\underline{H}^A = m\underline{\rho}^C \times \underline{v}^A + \underline{\underline{I}}^A \cdot \underline{\omega} \qquad (8.4)$$

***Proof*:**

Consider the rigid body shown;

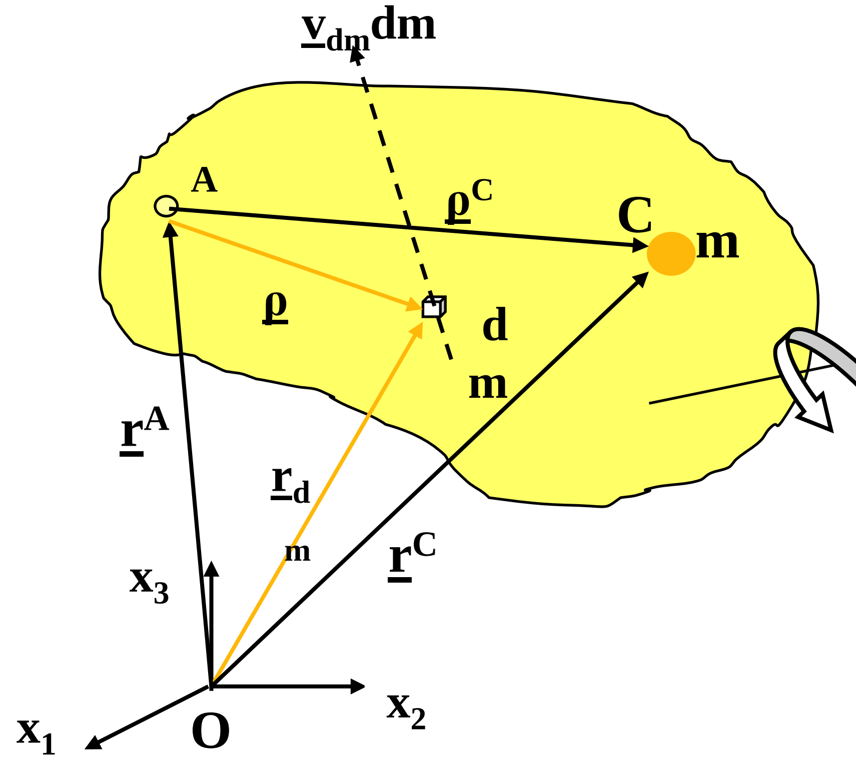
$\{x_i\}$**: its origin is located at the *fixed point* "O".**

For a rigid body, we have:

$$m = \int_m dm$$

$$\underline{P} = m\underline{v}^C = \int_m \underline{P}_{dm} = \int_m \underline{v}_{dm} dm$$

$$\underline{H}^A = \int_m \underline{H}^A_{dm} = \int_m \underline{\rho} \times \underline{v}_{dm} dm$$



but;

$$\underline{r}_{dm} = \underline{\rho} + \underline{r}^A \qquad and \qquad \underline{v}_{dm} = \dot{\underline{\rho}} + \underline{v}^A$$

$$\underline{H}^A = \int_m \underline{\rho} dm \times \underline{v}^A + \int_m \underline{\rho} \times \dot{\underline{\rho}} dm$$

and from the definition of mass center we have:

$m\underline{\rho}^{C} = \int_m \underline{\rho}\, dm$, **substituting results;**

$$\underline{H}^{A} = m\underline{\rho}^{C} \times \underline{v}^{A} + \int_m \underline{\rho} \times (\underline{\omega} \times \underline{\rho})\, dm \qquad \textbf{(8.5)}$$

Since $(\underline{\rho} = \underline{r}_{dm} - \underline{r}^{A}), or (\rho_i = x_i - x_i^{A})$ **is a constant magnitude vector, and the point "A" is a body point, then** $(\dot{\underline{\rho}} = \underline{\omega} \times \underline{\rho})$ **Now, the i-th component of the 2nd term in equation (8.5) may be expressed as:**

$$[\int_m \underline{\rho} \times (\underline{\omega} \times \underline{\rho})\, dm]_i = \omega_j \int_m [(x_k - x_k^{A})(x_k - x_k^{A})\delta_{ij} - (x_i - x_i^{A})(x_j - x_j^{A})]\, dm$$

$$= \quad I_{ij}^{A} \omega_j$$

Substituting the result into equation (8.5), we have:

$$\underline{H}^{A} = m\underline{\rho}^{C} \times \underline{v}^{A} + \underline{\underline{I}}^{A} \cdot \underline{\omega} \qquad \textbf{(8.4)}$$

However, if point "A" is selected such that

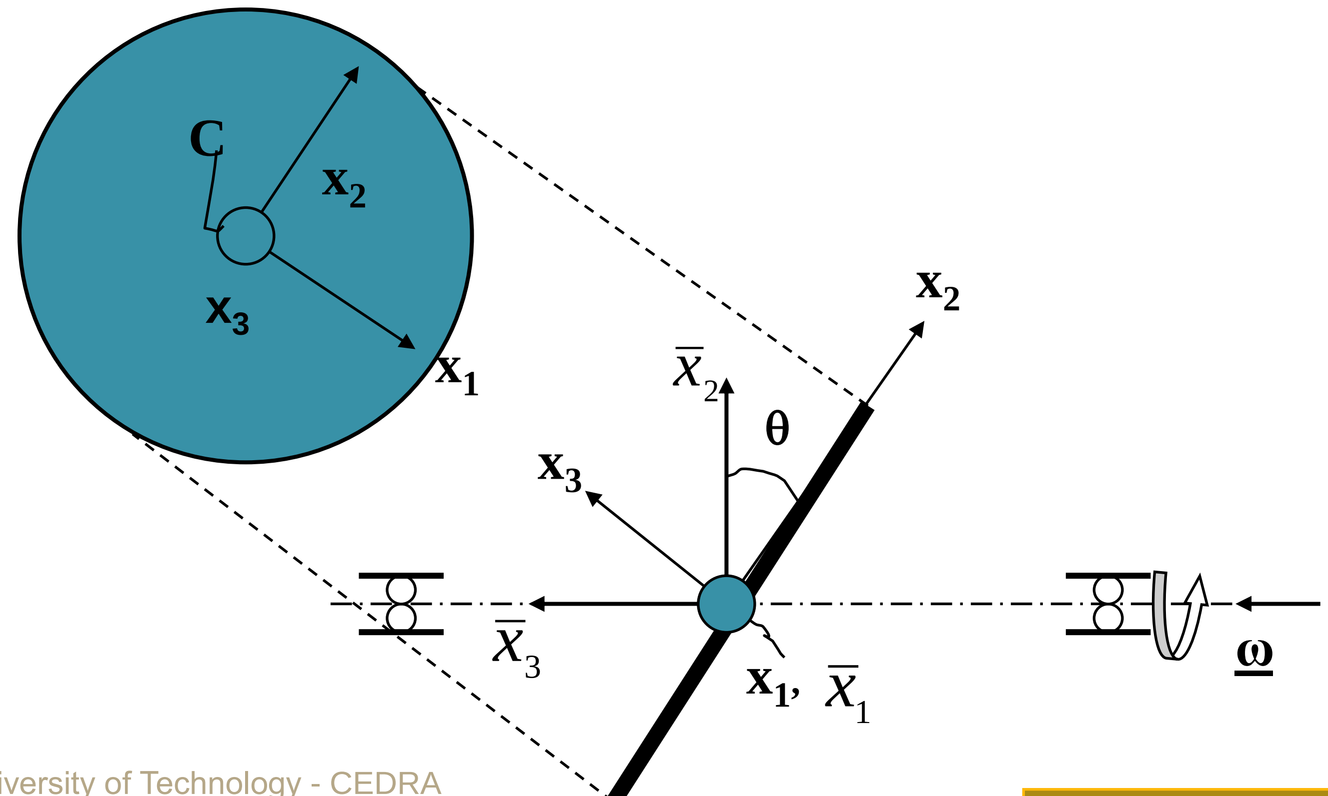
$$(\underline{v}^{A} = 0), \qquad or(A \equiv C), \quad or(\underline{\rho}^{C} \| \underline{v}^{A})$$

Then equation (8.4) reduces to:

$$\underline{H}^{A} = \underline{\underline{I}}^{A} \cdot \underline{\omega} \qquad or \qquad H_{i}^{A} = I_{ij}^{A}\omega_{j} \qquad \textbf{(8.6)}$$

Example: A thin disk of mass "m" and radius "R" is spinning about a fixed axis as shown. Determine the ***Central Moment of Momentum*** in terms of:

(a). the principal coordinates? (neglect the disk thickness).
(b). the coordinate with the rotating axis being as one of the coordinates?



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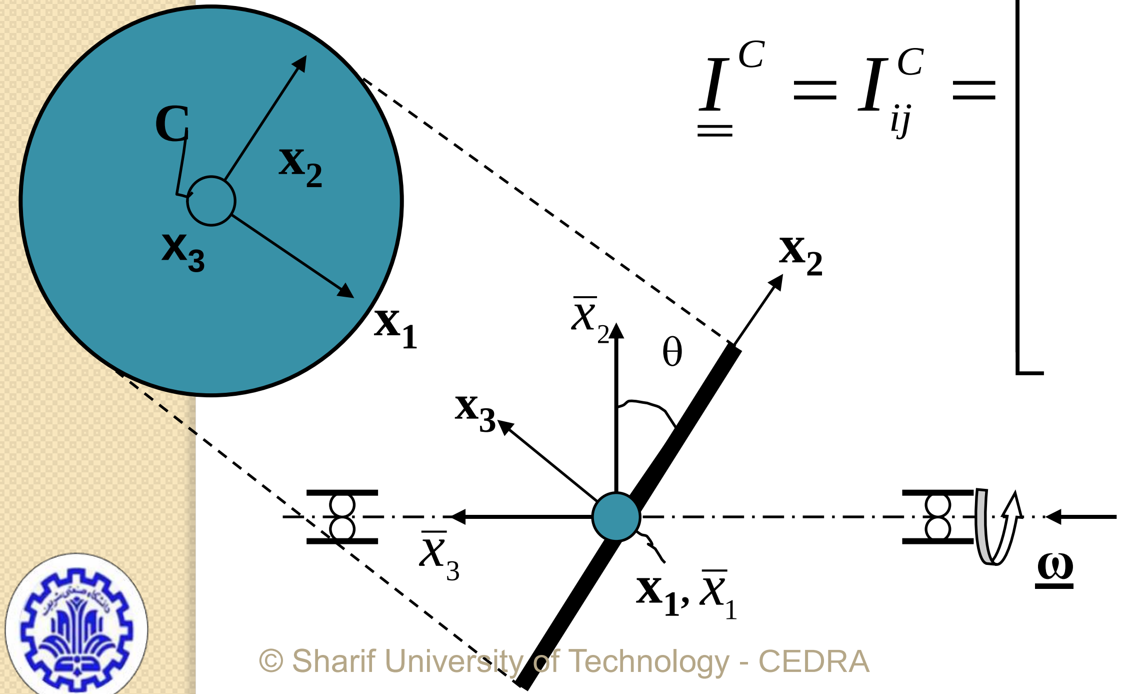
***Solution*:**

(a). In terms of *principal coordinates* $\{x_i\}$:

From symmetry, the coordinate $\{x_i\}$ is established in the principal directions;

$$\underline{\underline{I}}^C = I_{ij}^C = \begin{bmatrix} \frac{1}{4}mR^2 & 0 & 0 \\ 0 & \frac{1}{4}mR^2 & 0 \\ 0 & 0 & \frac{1}{2}mR^2 \end{bmatrix}$$



The angular velocity $\underline{\omega}$ in terms of principal coordinates is:

$$\{\omega_i\} = \begin{Bmatrix} 0 \\ -\omega\sin\theta \\ \omega\cos\theta \end{Bmatrix}$$ **, then;**

$$H_i^C = I_{ij}^C\omega_j = \begin{Bmatrix} 0 \\ -\frac{1}{4}mR^2\omega\sin\theta \\ \frac{1}{2}mR^2\omega\cos\theta \end{Bmatrix} \equiv \begin{Bmatrix} H_1^C \\ H_2^C \\ H_3^C \end{Bmatrix} = \underline{H}^C$$

(b). In terms of *coordinates oriented along the rotating axis* $\{\bar{X}_p\}$; Let us first setup the direction cosines such that:

$$\underline{\underline{T}} = \{\ell_{ip}\} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta & -s\theta \\ 0 & s\theta & c\theta \end{bmatrix}$$

Then, using the transform equation (7.25), we can write:

$\underline{\underline{\bar{I}}}^C = \underline{\underline{T}}^t \underline{\underline{I}}^C \underline{\underline{T}}$**, and**

$$\underline{\underline{\bar{I}}}^C = \begin{bmatrix} \frac{1}{4}mR^2 & 0 & 0 \\ 0 & \frac{1}{4}mR^2(1+\sin^2\theta) & \frac{1}{4}mR^2\sin\theta\cos\theta \\ 0 & \frac{1}{4}mR^2\sin\theta\cos\theta & \frac{1}{4}mR^2(1+\cos^2\theta) \end{bmatrix}$$

Then, the ***Central Moment of Momentum*** **is:**

$$\underline{\overline{H}}^{C} = \underline{\underline{\overline{I}}}^{C} \cdot \underline{\overline{\omega}}$$; **where** $$\underline{\overline{\omega}} = \begin{Bmatrix} 0 \\ 0 \\ \omega \end{Bmatrix}$$, **and;**

$$\underline{\overline{H}}^{C} = \begin{Bmatrix} \overline{H}_1^{C} \\ \overline{H}_2^{C} \\ \overline{H}_3^{C} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \frac{1}{4} mR^2 \omega \sin\theta \cos\theta \\ \frac{1}{4} mR^2 \omega (1 + \cos^2\theta) \end{Bmatrix}$$

Observations :

- Although the spin axis is fixed, the moment of momentum has components along the other axes.

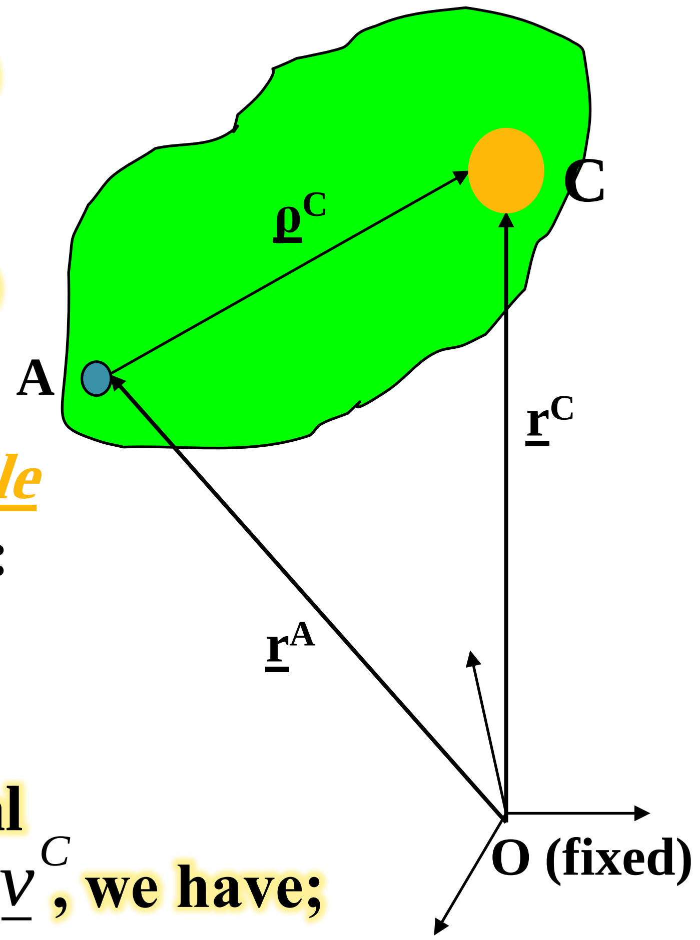
- Even when "ω" is constant, while the moment of momentum ***components*** may be ***constant*** in the body coordinate $\{x_i\}$, the moment of momentum ***vector*** will rotate about the rotation axis at a speed of "ω".

- The time rate of change of moment of momentum due to the change in direction is an indication of presence of an exterior moment on the body.

***Kinetic Principles of a Rigid Body*:** **Consider a rigid body as shown, where point "A" is a body point, then:**

Momentum Principle **(P-Principle)** **for the Rigid Body is:**

$$\underline{f} = m\underline{a}^{C} = \dot{\underline{P}} \qquad (8.7)$$



Momentum of Momentum Principle **(H-Principle) for the Rigid Body is:**

In Chapter-6 we showed that for a system of particles, with the general moment center "A", and $\underline{P} = m\underline{v}^{C}$**, we have;**

$$\underline{M}^{A} = \dot{\underline{H}}^{A} + (\underline{v}^{A} \times \underline{P}) \qquad (6.23)$$

However, if

$$\left\{\begin{array}{l} \underline{v}^A = 0 \quad or \quad ("A"-is-a-fixed-point) \\ A \equiv C \\ \underline{v}^A \,||\, \underline{P} \end{array}\right\} \Rightarrow \underline{M}^A = \underline{\dot{H}}^A$$

Now for a rigid body, the moment center "A" is also a *body point*, hence:

$$\underline{H}^A = \underline{\rho}^C \times m\underline{v}^A + \underline{\underline{I}}^A \cdot \underline{\omega} \equiv (\underline{r}^C - \underline{r}^A) \times m\underline{v}^A + \underline{\underline{I}}^A \cdot \underline{\omega}$$

, and;

$$\underline{\dot{H}}^A = (\underline{v}^C - \underline{v}^A) \times m\underline{v}^A + (\underline{r}^C - \underline{r}^A) \times m\underline{a}^A + \frac{d}{dt}(\underline{\underline{I}}^A \cdot \underline{\omega})$$

(8.8)

However, if

$$\left\{\begin{matrix} "A": \quad is-a-fixed-point, \quad or \\ A \equiv C, \quad or \\ \underline{v}^A = 0 \quad or \quad \underline{v}^A \,\|\, \underline{v}^C, and \quad \underline{\rho}^C \,\|\, \underline{a}^A \end{matrix}\right\} \Rightarrow \underline{M}^A = \dot{\underline{H}}^A = \frac{d}{dt}(\underline{\underline{I}}^A \cdot \underline{\omega})$$

Then:

$$\underline{M}^A = \dot{\underline{H}}^A = \frac{d}{dt}(\underline{\underline{I}}^A \cdot \underline{\omega}), \quad or \quad M_i^A = \frac{d}{dt}(I_{ij}^A \omega_j) \quad \textbf{(8.9)}$$

Equation (8.9) is referred to as the *Euler's Equation*.

***Definition*: A moment center satisfying the conditions stated above is called an *Eulerian Moment Center*.**

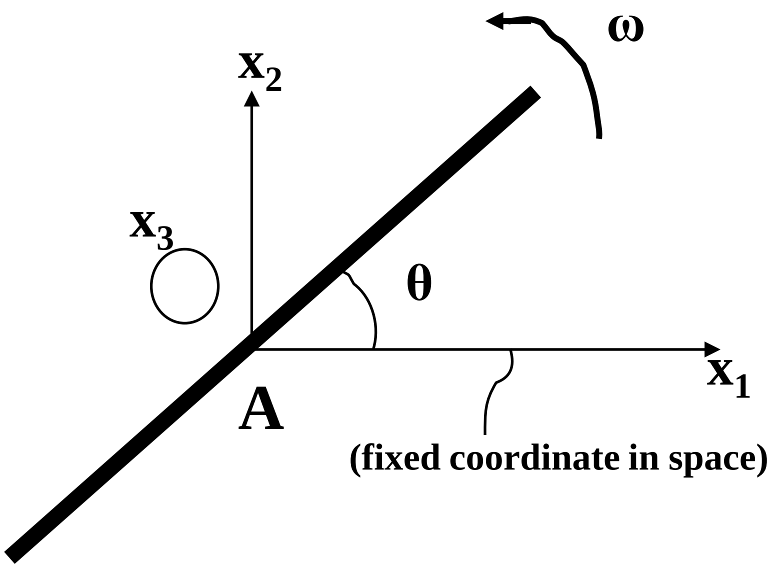
Euler's Equation as a function of coordinate variables :

$$M_i^A = \frac{d}{dt}[I_{ij}^A(x)\omega_j]; \quad \underline{for - fixed} \{x_i\} \quad \textbf{where:} \qquad \textbf{(8.10)}$$

$I_{ij}^A(x)$ **: are components of inertia tensor (function of coordinate variables)**

Example: ***A Thin Uniform Rod***:

$$\underset{=\alpha}{I^A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{ml^2}{12} & 0 \\ 0 & 0 & \frac{ml^2}{12} \end{bmatrix}$$



(i.e. $I_{11}^A(\theta), I_{12}^A(\theta), \ldots$**),.where:**

$$I_{ij}^{A}(\theta) = \underline{\underline{T}}^{t}\,\underline{\underline{I}}_{\alpha}^{A}\,\underline{\underline{T}} = \begin{bmatrix} C\theta & S\theta & 0 \\ -S\theta & C\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{m\ell^2}{12} & 0 \\ 0 & 0 & \frac{m\ell^2}{12} \end{bmatrix} \begin{bmatrix} C\theta & -S\theta & 0 \\ S\theta & C\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{m\ell^2}{12}S^2\theta & \frac{m\ell^2}{12}S\theta C\theta & 0 \\ \frac{m\ell^2}{12}S\theta C\theta & \frac{m\ell^2}{12}C^2\theta & 0 \\ 0 & 0 & \frac{m\ell^2}{12} \end{bmatrix} = \frac{m\ell^2}{12}\begin{bmatrix} S^2\theta & S\theta C\theta & 0 \\ S\theta C\theta & C^2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore; $I_{11}^{A}(\theta) = \frac{1}{12} m\ell^2 Sin^2\theta$, **and etc..** ***Euler's Equation***

in this form is *coordinate dependent*.

متشکرم

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