بسم الله الرحمن الرحیم

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INERTIA TENSOR

***Purpose*:**

- **To Study Inertia Tensor of a Material System.**

***Topics*:**

- **Inertia Tensor of a Mass Particle.**
- **Inertia Tensor of a System of Particles.**
- **Inertia Tensor of a Continuum.**
- **Transfer Theorem.**
- **Principal Values of Inertia Tensor.**

From Chapter-6, we defined the ***Kinetic States*** as:

- **For a Single particle;**

Linear Momentum (Momentum):

$$\underline{P} = m\underline{v} \quad or \quad P_i = mv_i \tag{7.1}$$

Moment of Momentum (Angular Momentum):

$$\underline{H}^O = \underline{r} \times \underline{P} = m\underline{r} \times \underline{\dot{r}}, \quad or$$

$$H_i^O = \gamma_{ijk} x_j P_k \tag{7.2}$$

, where: "O" is a moment center.

❖ For a System of Particles;

***Linear Momentum (Momentum)*:**

$$\underline{P} = \sum_{\beta=1}^{N} \underline{P}^{\beta} = \sum_{\beta=1}^{N} m_{\beta} \underline{v}^{\beta}, \quad or$$

$$P_i = \sum_{\beta=1}^{N} P_i^{\beta} \qquad (i = 1,2,3)$$

(7.3)

***Moment of Momentum (Angular Momentum)*:**

$$\underline{H}^{O} = \sum_{\beta=1}^{N} \underline{H}_{\beta}^{O} = \sum_{\beta=1}^{N} \underline{r}^{\beta} \times \underline{P}^{\beta} = \sum_{\beta=1}^{N} m_{\beta} \underline{r}^{\beta} \times \underline{v}^{\beta}, \quad or$$

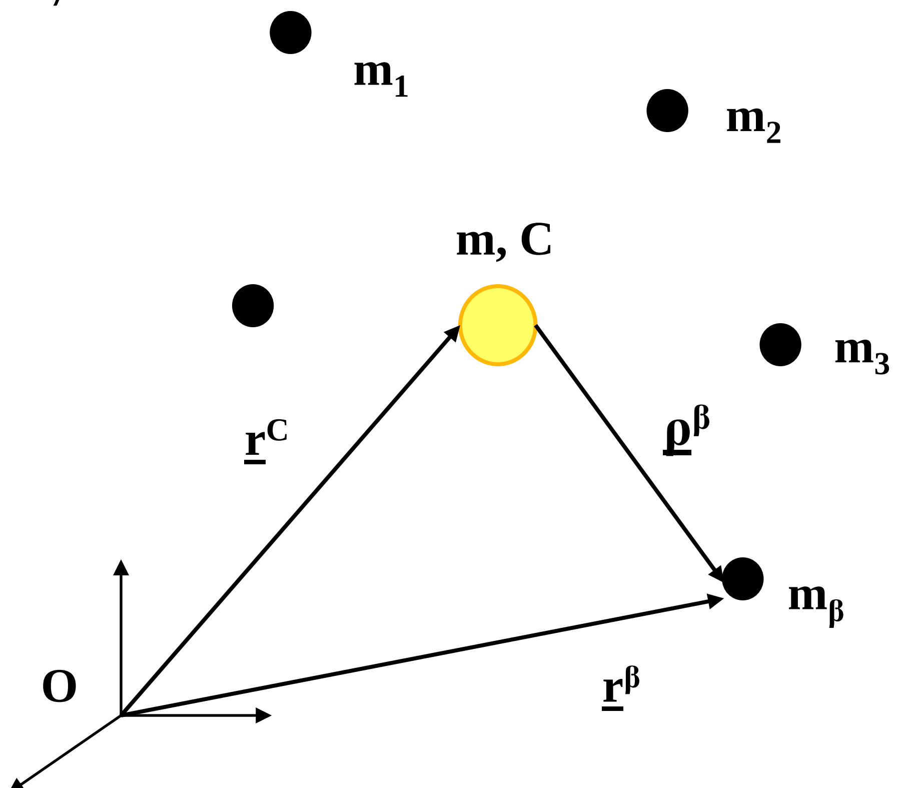
$$H_i^{\beta} = \sum_{\beta=1}^{N} \gamma_{ijk} x_j^{\beta} P_k^{\beta} \qquad (i, j, k \equiv 1,2,3)$$

(7.4)

Where; $\underline{r}^{\beta}$: position vector of the "βth" particle from the moment center "O".

or, by Theorem-16, for the equivalent mass system, the Total/Global Momentum and Global Moment of Momentum of the system of particles is:

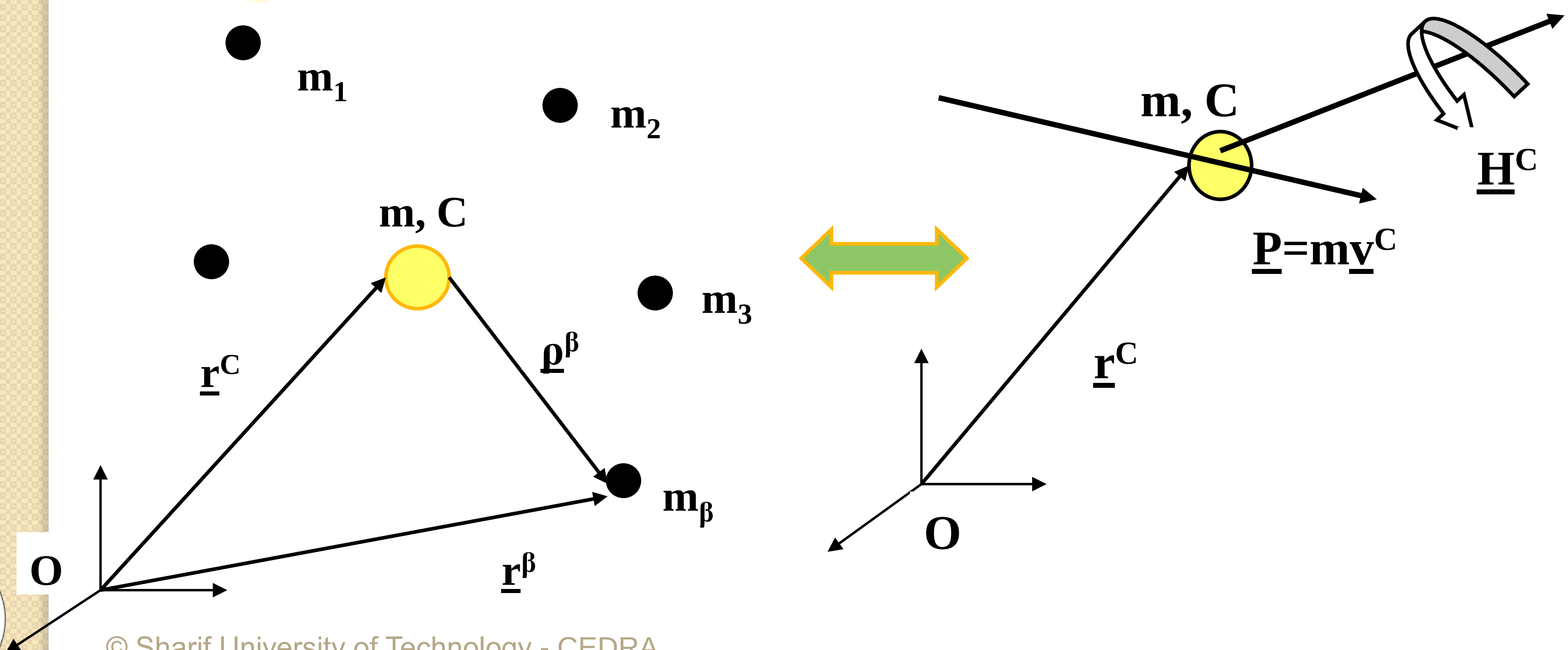
$$\underline{P} = m\underline{v}^{C} = \sum_{\beta=1}^{N} m_{\beta}\underline{v}^{\beta} = (\text{Global Momentum}) \qquad (7.5)$$



$$\underline{H}^O = \underline{H}^C + \underline{r}^C \times \underline{P} = \underline{H}^C + \underline{H}^O_{eq} \qquad \text{(Global M.O.M.)} \qquad (7.6)$$

Where;

- $\underline{H}^O$ **: (M.O.M. about any point "O" in space).**
- $\underline{H}^C = \sum_{\beta=1}^{N} \underline{\rho}^\beta \times \underline{P}^\beta = \sum_{\beta=1}^{N} \underline{\rho}^\beta \times m_\beta \dot{\underline{\rho}}^\beta$ **(Central M.O.M.)**
- $\underline{H}^O_{eq} = \underline{r}^C \times \underline{P}$



Now, let us consider the system of particles shown.

We wish to define the quantity $\underline{H}^{C} = ?$ **(Ce ntral M.O.M.).**

To do this, suppose that the system is:

- *Locked rigidly*, so that the distance between any pair of particles remain constant throughout the dynamic process.

***Therefore*: The system's rotation may be described by a single angular velocity "ω".**

$$\underline{H}^{C} = \sum_{\beta=1}^{N} \underline{\rho}^{\beta} \times \underline{P}^{\beta} = \sum_{\beta=1}^{N} \underline{\rho}^{\beta} \times m_{\beta} \dot{\underline{\rho}}^{\beta} \qquad where: \begin{Bmatrix} |\underline{\rho}| \equiv const. \\ \{m_{\beta}\} \equiv rigid \\ C \equiv Body - Point \end{Bmatrix}$$

$$\underline{\rho}^{\beta} = \bar{x}_i^{\beta} \underline{u}_i$$ **; where** $\{\bar{x}_i\}$**: its origin is at the mass center.**

$$\underline{\omega}_{\beta} = \underline{\omega}$$

Therefore: $$\dot{\underline{\rho}}^{\beta} = \underline{\omega} \times \underline{\rho}^{\beta}$$

***Hence*:**

$$\underline{H}^{C} = \sum_{\beta=1}^{N} m_{\beta} \underline{\rho}^{\beta} \times (\underline{\omega} \times \underline{\rho}^{\beta}) \quad or \quad H_i^{C} = [\sum_{\beta=1}^{N} m_{\beta} \underline{\rho}^{\beta} \times (\underline{\omega} \times \underline{\rho}^{\beta})]_i$$

But,

$$\underline{\rho}^{\beta} \times (\underline{\omega} \times \underline{\rho}^{\beta}) = (\underline{\rho}^{\beta} \cdot \underline{\rho}^{\beta})\underline{\omega} - (\underline{\rho}^{\beta} \cdot \underline{\omega})\underline{\rho}^{\beta}$$

$$[\underline{\rho}^{\beta} \times (\underline{\omega} \times \underline{\rho}^{\beta})]_i = (\underline{\rho}^{\beta} \cdot \underline{\rho}^{\beta})\omega_i - (\underline{\rho}^{\beta} \cdot \underline{\omega})\bar{x}_i^{\beta}$$

$$= (\bar{x}_k^{\beta}\, \bar{x}_k^{\beta})\omega_j \delta_{ij} - (\bar{x}_j^{\beta} \omega_j)\bar{x}_i^{\beta}$$

$$= (\bar{x}_k^{\beta} \bar{x}_k^{\beta}\, \delta_{ij} - \bar{x}_i^{\beta} \bar{x}_j^{\beta})\omega_j$$

***Therefore*:**

$$H_i^C = [\sum_{\beta=1}^{N} m_{\beta} (\bar{x}_k^{\beta} \bar{x}_k^{\beta} \delta_{ij} - \bar{x}_i^{\beta} \bar{x}_j^{\beta})]\omega_j$$

$$= I_{ij}^C \omega_j \qquad \textbf{(7.7)}$$

$$\underline{H}^C = \underline{\underline{I}}^C \cdot \underline{\omega}$$

where:

$$\underline{\underline{I}}^C \equiv I_{ij}^C = [\sum_{\beta=1}^{N} m_\beta (\bar{x}_k^\beta \bar{x}_k^\beta \delta_{ij} - \bar{x}_i^\beta \bar{x}_j^\beta)] \qquad \textbf{(7.8)}$$

≡ (*The Inertia Tensor*; a 2nd order and symmetrical tensor), or:

$$I_{ij}^C \equiv \begin{bmatrix} I_{11}^C & I_{12}^C & I_{13}^C \\ I_{21}^C & I_{22}^C & I_{23}^C \\ I_{31}^C & I_{32}^C & I_{33}^C \end{bmatrix}$$

Compare (for a rigid system): $\left\{ \begin{array}{c} \underline{H}^C = \underline{\underline{I}}^C \cdot \underline{\omega} \\ \underline{P} = m\underline{v}^C \end{array} \right\}$ **where:**

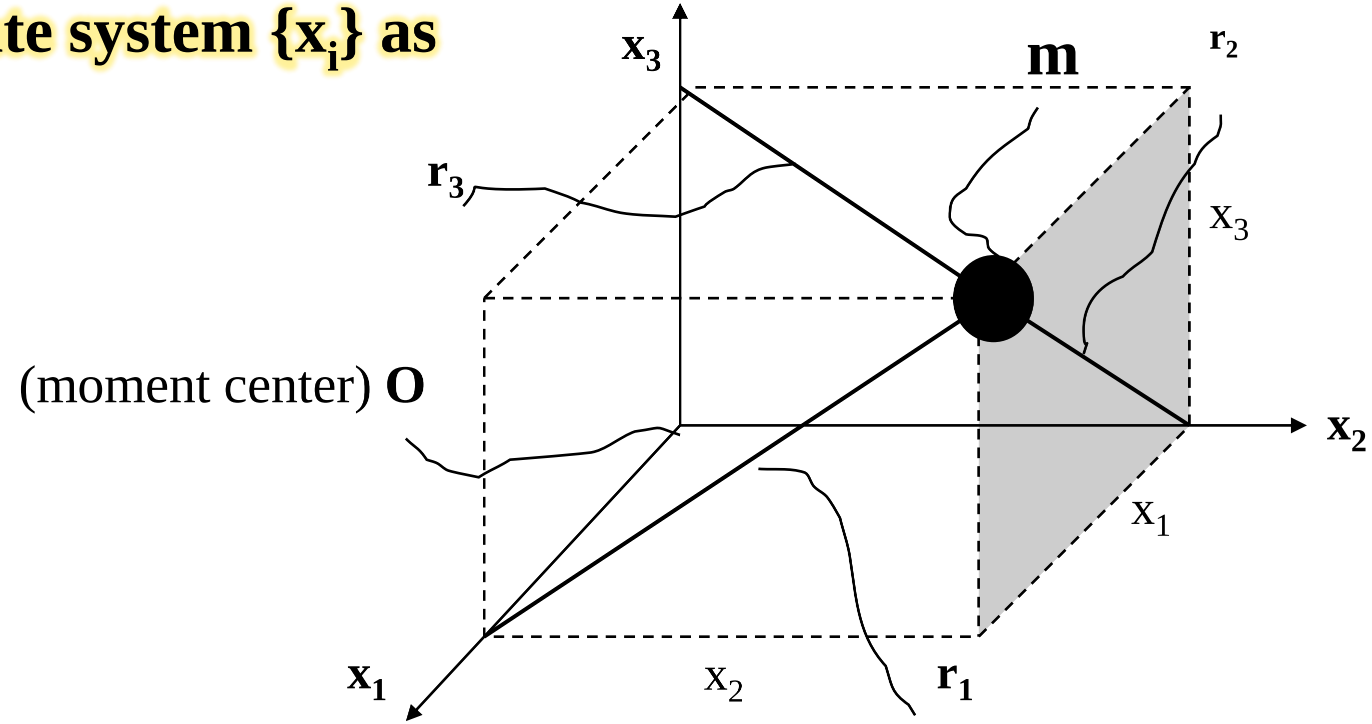
$\underline{\omega}$ *and* $\underline{v}^C$ **: are the velocity properties, and**

$\underline{\underline{I}}^C$ **: plays the role of mass "m" in describing the kinetic state of a rigid system.**

Definition: The "***Inertia Tensor***" actually defines the mass distribution of a material system.

Inertia Tensor of a Particle of Mass "m" about a fixed point "O":

Consider the particle "m" in coordinate system $\{x_i\}$ as shown;



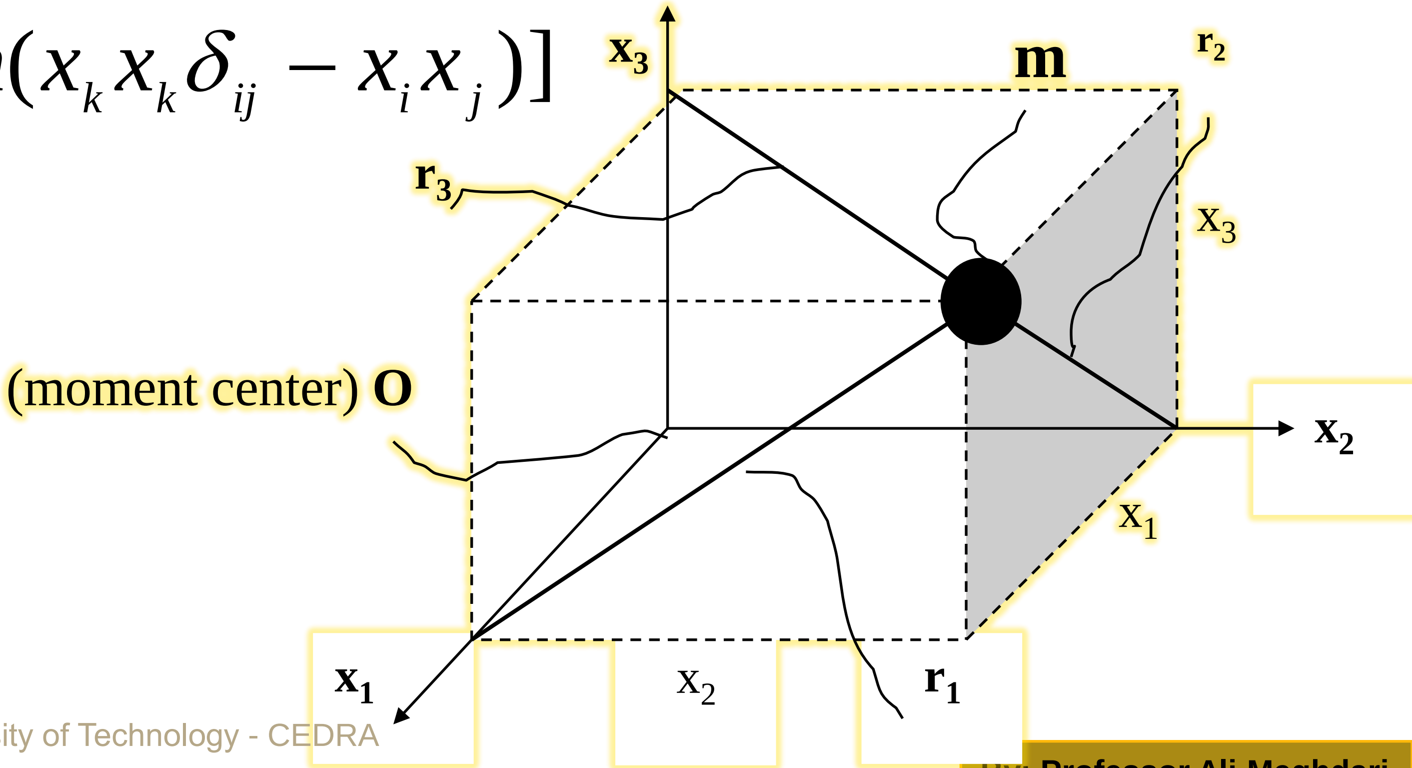
$$\underline{H}^{O} = \underline{r} \times \underline{P} = \underline{r} \times m\underline{v} = \underline{r} \times m\underline{\dot{r}}$$

$$= m\underline{r} \times (\underline{\omega} \times \underline{r})$$

$$= m(\underline{r} \cdot \underline{r})\underline{\omega} - (\underline{r} \cdot \underline{\omega})\underline{r}$$

$$H_i^{O} = [m(x_k x_k \delta_{ij} - x_i x_j)]\omega_j \qquad \textbf{(7.9)}$$

$$I_{ij}^{O} = [m(x_k x_k \delta_{ij} - x_i x_j)]$$



$$I_{ij}^{O} = [m(x_k x_k \delta_{ij} - x_i x_j)] \qquad \textbf{(7.9)}$$

Definition: **The *moments of inertia* of the particle "m" about the axes $\{x_i\}$ are:**

$$I_{11}^{O} = m(x_1^2 + x_2^2 + x_3^2 - x_1^2) = m(x_2^2 + x_3^2) = mr_1^2 \qquad for \quad i=1, j=1$$

$$I_{22}^{O} = m(x_1^2 + x_2^2 + x_3^2 - x_2^2) = m(x_1^2 + x_3^2) = mr_2^2 \qquad for \quad i=2, j=2$$

$$I_{33}^{O} = m(x_1^2 + x_2^2 + x_3^2 - x_3^2) = m(x_1^2 + x_2^2) = mr_3^2 \qquad for \quad i=3, j=3$$

Definition: **The *products of inertia* of the particle "m" are:**

$$I_{12}^{O} = m(0 - x_1 x_2) = -m x_1 x_2 \qquad for \quad i=1, j=2$$

$$I_{23}^{O} = m(0 - x_2 x_3) = -m x_2 x_3 \qquad for \quad i=2, j=3$$

$$I_{31}^{O} = m(0 - x_3 x_1) = -m x_3 x_1 \qquad for \quad i=3, j=1$$

Inertia Tensor of a System of Particles about a fixed point "O": A physical property of the material system;

$$I_{ij}^{O} = \sum_{\beta=1}^{N} m_{\beta}[x_k^{\beta} x_k^{\beta} \delta_{ij} - x_i^{\beta} x_j^{\beta}] \qquad \textbf{(7.10)}$$

$$I_{ij}^{O} = \sum_{\beta=1}^{N} m_{\beta} \begin{bmatrix} (x_2^{\beta})^2 + (x_3^{\beta})^2 & -x_1^{\beta} x_2^{\beta} & -x_1^{\beta} x_3^{\beta} \\ -x_2^{\beta} x_1^{\beta} & (x_3^{\beta})^2 + (x_1^{\beta})^2 & -x_2^{\beta} x_3^{\beta} \\ -x_3^{\beta} x_1^{\beta} & -x_3^{\beta} x_2^{\beta} & (x_1^{\beta})^2 + (x_2^{\beta})^2 \end{bmatrix}$$

***Note*: To write the inertia tensor about an *arbitrary* point "A" in space, simply do a coordinate shift, as:**

$$I_{ij}^{A} = \sum_{\beta=1}^{N} m_{\beta}[(x_k^{\beta} - x_k^{A})(x_k^{\beta} - x_k^{A})\delta_{ij} - (x_i^{\beta} - x_i^{A})(x_j^{\beta} - x_j^{A})]$$

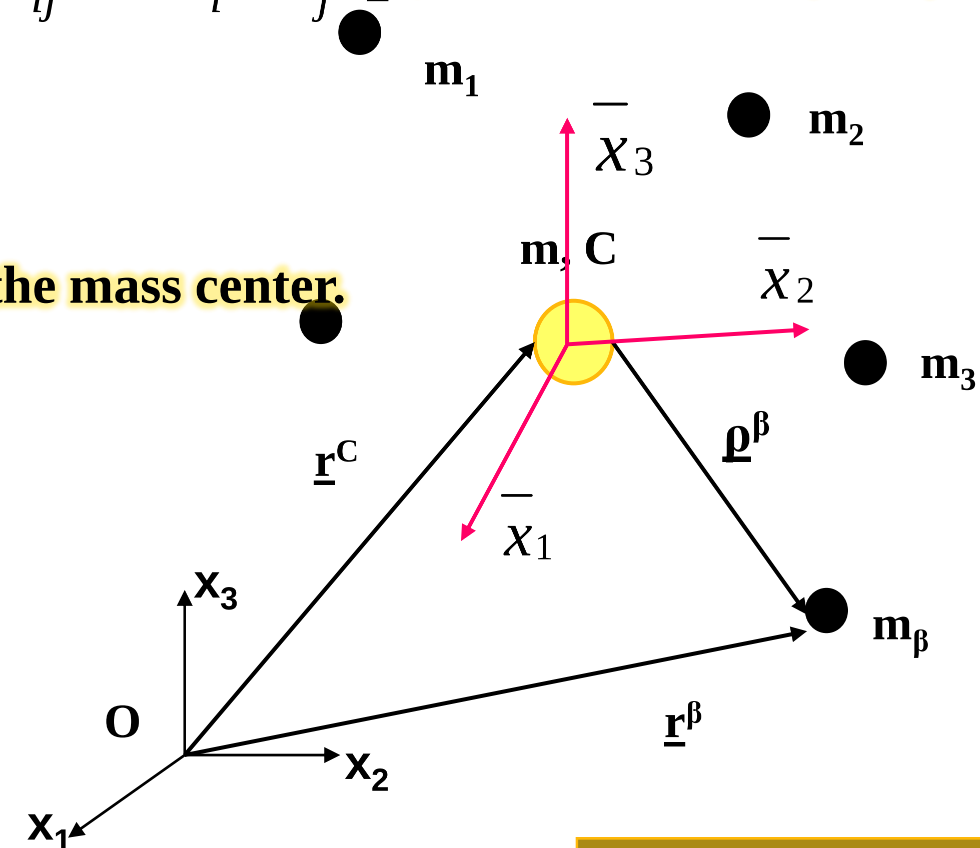
(7.11)

Definition: **When the moment center is specifically the mass center, the inertia tensor is called the *Central Inertia Tensor*, "** $\underline{\underline{I}}^{C}$ **", where:**

$$I_{ij}^{C} = \sum_{\beta=1}^{N} m_{\beta}[\bar{x}_k^{\beta}\bar{x}_k^{\beta}\delta_{ij} - \bar{x}_i^{\beta}\bar{x}_j^{\beta}]\,; \qquad \textbf{(7.12)}$$

Where:

$\{\bar{x}_i\}$**: its origin is located at the mass center.**



***Inertia Tensor for a Continuum*:** **A generalization of the *inertia tensor for a system of particles* where the finite sums " $\sum$ " are integrally summed " $\int$ ".**

$$I_{ij}^{O} = \int_{m} (x_k x_k \delta_{ij} - x_i x_j) dm \quad (7.13)$$

Ex:

$$\left\{ \begin{array}{l} I_{11}^{O} = \int_{m} (x_2^2 + x_3^2) dm \\ I_{12}^{O} = -\int_{m} x_1 x_2 dm \end{array} \right\}$$

And about any *arbitrary moment center* like "A" is:

$$I_{ij}^{A} = \int_{m} [(x_k - x_k^A)(x_k - x_k^A)\delta_{ij} - (x_i - x_i^A)(x_j - x_j^A)] dm$$

(7.14)

***Transfer Theorem*-(18): When the mass, the mass center, and the central inertia tensor are known, the inertia tensor of a mass system about any given moment center can be obtained from:**

$$I_{ij}^{A} = m[(x_k^C - x_k^A)(x_k^C - x_k^A)\delta_{ij} - (x_i^C - x_i^A)(x_j^C - x_j^A)] + I_{ij}^{C}$$

$$\underline{\underline{I}}^{A} = \underline{\underline{I}}_{eq.}^{A} + \underline{\underline{I}}^{C} \qquad \textbf{(7.15)}$$

where; $\underline{\underline{I}}_{eq.}^{A}$: inertia tensor about "A" due to a single particle of mass "m" at the mass center.

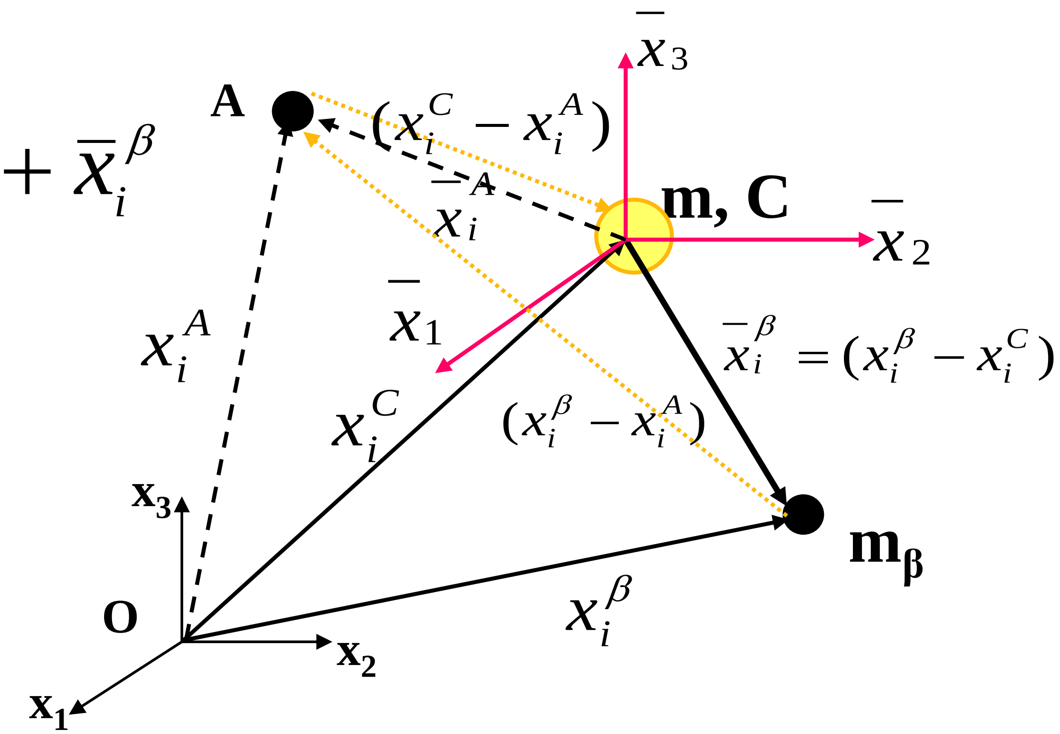
Proof:

$\{\bar{x}_i\}$: **its origin is located at the mass center.**

$\{x_i\}$: **its origin is located at the fixed point "O".**

$$x_i^{\beta} = x_i^{C} + \bar{x}_i^{\beta}$$

$$x_i^{\beta} - x_i^{A} = x_i^{C} - x_i^{A} + \bar{x}_i^{\beta}$$



$$(x_i^{\beta} - x_i^{A}) = (x_i^{C} - x_i^{A}) + (x_i^{\beta} - x_i^{C})$$

$$(x_i^\beta - x_i^A) = (x_i^C - x_i^A) + (x_i^\beta - x_i^C)$$

Substituting the last equation into the Equation (7.11) and simplifying results in:

***Note*: *Inertia Tensor* about an *arbitrary* point "A" in space, can be computed by a coordinate shift, as:**

$$I_{ij}^A = \sum_{\beta=1}^{N} m_\beta [(x_k^\beta - x_k^A)(x_k^\beta - x_k^A)\delta_{ij} - (x_i^\beta - x_i^A)(x_j^\beta - x_j^A)]$$

(7.11)

$$I_{ij}^A = m[(x_k^C - x_k^A)(x_k^C - x_k^A)\delta_{ij} - (x_i^C - x_i^A)(x_j^C - x_j^A)] + I_{ij}^C$$

(7.15)

However, if the origin "O" is selected as the moment center, then $x_i^A = 0$, and Equation (7.15) reduces to:

$$\begin{cases} I_{ij}^{O} = m(x_k^C x_k^C \delta_{ij} - x_i^C x_j^C) + I_{ij}^{C} \\ \underline{\underline{I}}^{O} = \underline{\underline{I}}_{eq.}^{O} + \underline{\underline{I}}^{C} \end{cases} \quad (7.16)$$

or:

$$\underline{\underline{I}}^{O} = \underline{\underline{I}}^{C} + m\begin{bmatrix} (x_2^C)^2 + (x_3^C)^2 & -x_1^C x_2^C & -x_1^C x_3^C \\ -x_2^C x_1^C & (x_3^C)^2 + (x_1^C)^2 & -x_2^C x_3^C \\ -x_3^C x_1^C & -x_3^C x_2^C & (x_1^C)^2 + (x_2^C)^2 \end{bmatrix}$$

(7.17)

***Principal Values of the Inertia Tensor*:** **In rigid body dynamics, it is often convenient to use a coordinate system fixed to the rigid body in which the products of inertial are zero. In this case, the inertial tensor "** $\underline{\underline{I}}$ **" will be a *diagonal matrix* such as:**

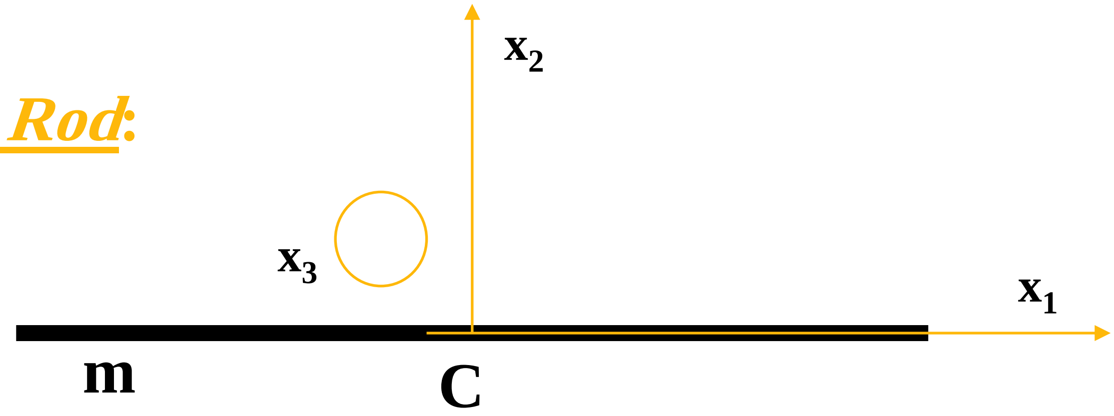
$$I_{\alpha} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \qquad (7.18)$$

This coordinate system is then called the *Principal Coordinate*, and the moments of inertia about the principal axes are called the *Principal Moments of Inertia*, and the three planes formed by the principal axes are called the *Principal Planes*.

Example:

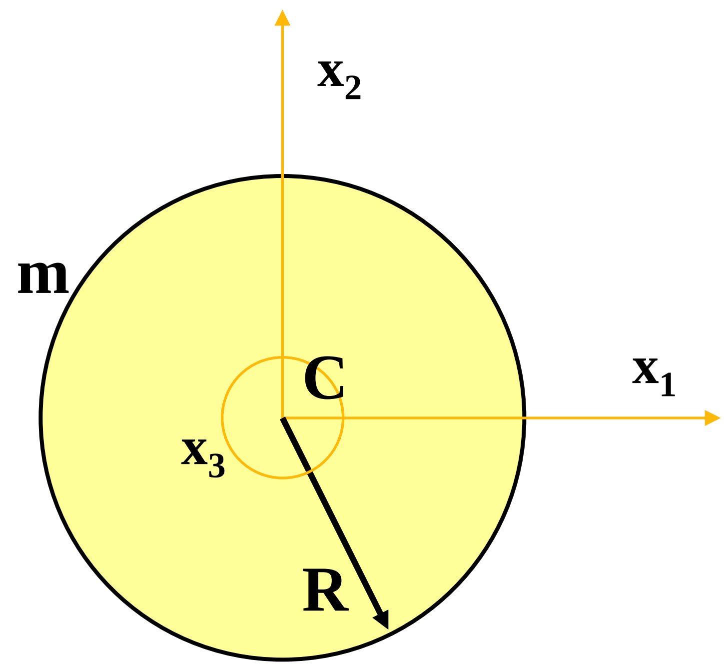
- ***A Thin Uniform Rod*:**

$$\underline{\underline{I}}^C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{ml^2}{12} & 0 \\ 0 & 0 & \frac{ml^2}{12} \end{bmatrix}$$



- ***A Thin Uniform Disk*:**

$$\underline{\underline{I}}^C = \begin{bmatrix} \frac{1}{4}mR^2 & 0 & 0 \\ 0 & \frac{1}{4}mR^2 & 0 \\ 0 & 0 & \frac{1}{2}mR^2 \end{bmatrix}$$



➢ *A Sphere of Radius "R" and Mass "m"*:

$$\underline{\underline{I}}^{C} = \frac{2}{5} m R^{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

***More on the Principal Inertia Tensor*:** **If we examine the moment and product of inertia terms for all possible orientation of the coordinate axes with respect to a rigid body for a given origin, we will find in the general case *one unique orientation* "$\{x_\alpha\}$" for which the products of inertia are zero.**

Theorem-19: Given "$\underline{\underline{I}}^{O} = I_{ij}^{O}$", there exist three principal values of "$\underline{\underline{I}}^{O}$", namely I_{α}^{O} (*Principal Moments of Inertial*) which are the *eigenvalues* of the inertia tensor I_{ij}^{O}, and may be computed as follows:

$$\left| I_{ij}^{O} - I_{\alpha}^{O} \underline{\underline{E}} \right| = 0$$

, and let $I_{\alpha}^{O} \equiv I$ ($\underline{\underline{E}}$ = ***Unit Matrix*** $= \delta_{ij}$)

therefore; **(7.19)**

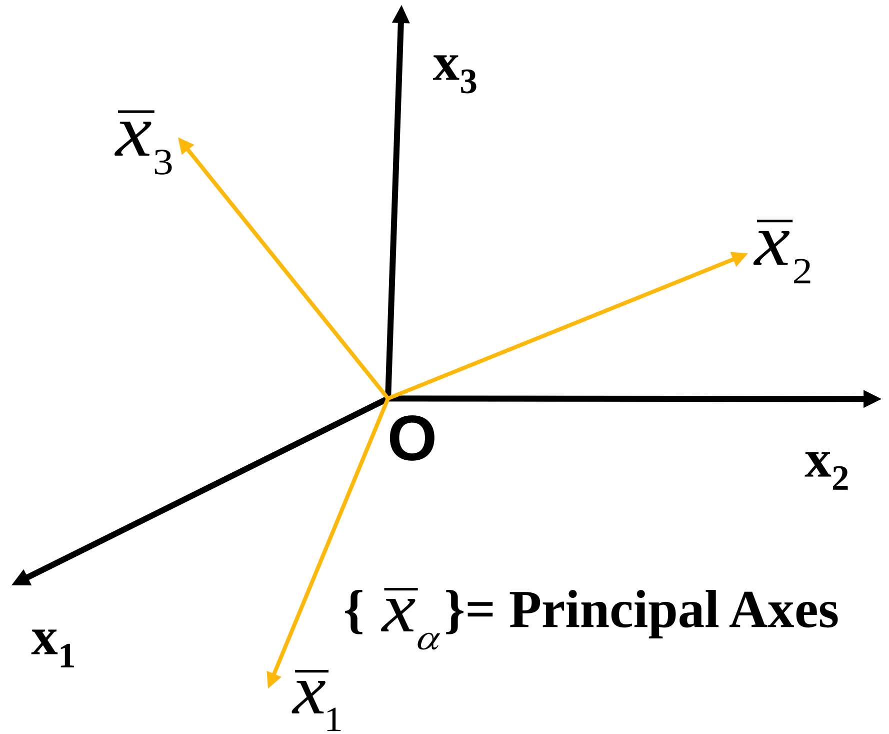
$$I^3 - J_1 I^2 - J_2 I - J_3 = 0 \qquad (Characteristic - Equation), \qquad where:$$

$$J_1 = I_{ii}^{O}$$

$$J_2 = \frac{1}{2}(I_{ij}^{O} I_{ij}^{O} - I_{ii}^{O} I_{jj}^{O}) \qquad \textbf{(7.20)}$$

$$J_3 = \left| I_{ij}^{O} \right|$$

and the direction cosines of the principal axes of inertia $\{\ell_{i\alpha}\}$ are the corresponding ***normalized eigenvectors***, defined as follows:



$$(I_{ij}^{O} - I_{\alpha}^{O}\delta_{ij})\ell_{i\alpha} = 0$$

$$\ell_{i\alpha}\ell_{i\alpha} = 1$$

(7.21)

Example: For the $\bar{X}_1$ - axis we can write:

$$\begin{Bmatrix} (I_{11}^{O}-I_{1}^{O})\ell_{1\bar{1}}+I_{21}^{O}\ell_{2\bar{1}}+I_{31}^{O}\ell_{3\bar{1}}=0 \\ I_{12}^{O}\ell_{1\bar{1}}+(I_{22}^{O}-I_{1}^{O})\ell_{2\bar{1}}+I_{32}^{O}\ell_{3\bar{1}}=0 \\ I_{13}^{O}\ell_{1\bar{1}}+I_{23}^{O}\ell_{2\bar{1}}+(I_{33}^{O}-I_{1}^{O})\ell_{3\bar{1}}=0 \\ \ell_{1\bar{1}}^{2}+\ell_{2\bar{1}}^{2}+\ell_{3\bar{1}}^{2}=1 \end{Bmatrix} \qquad (7.22)$$

From equations (7.22), we can write:

$$\underline{\bar{e}}_1 \equiv \begin{Bmatrix} \ell_{1\bar{1}}\underline{e}_1 \\ \ell_{2\bar{1}}\underline{e}_2 \\ \ell_{3\bar{1}}\underline{e}_3 \end{Bmatrix}$$

(In the first 3 Equations of (7.22), only 2 are independent, and the 3rd is a linear combination of the others.)

And similarly for $\bar{x}_2$ and $\bar{x}_3$ axes we have:

$$\underline{\bar{e}}_2 \equiv \begin{Bmatrix} \ell_{1\bar{2}}\,\underline{e}_1 \\ \ell_{2\bar{2}}\,\underline{e}_2 \\ \ell_{3\bar{2}}\,\underline{e}_3 \end{Bmatrix}, \quad \underline{\bar{e}}_3 \equiv \begin{Bmatrix} \ell_{1\bar{3}}\,\underline{e}_1 \\ \ell_{2\bar{3}}\,\underline{e}_2 \\ \ell_{3\bar{3}}\,\underline{e}_3 \end{Bmatrix}$$

, and therefore:

$$\{\underline{\bar{e}}_\alpha\} \equiv \begin{bmatrix} \ell_{1\bar{1}} & \ell_{2\bar{1}} & \ell_{3\bar{1}} \\ \ell_{1\bar{2}} & \ell_{2\bar{2}} & \ell_{3\bar{2}} \\ \ell_{1\bar{3}} & \ell_{2\bar{3}} & \ell_{3\bar{3}} \end{bmatrix} \{\underline{e}_i\} \equiv \underline{\underline{T}}\{\underline{e}_i\}, \qquad or$$

(7.23)

$$\{\underline{e}_i\} \equiv \begin{bmatrix} \ell_{1\bar{1}} & \ell_{1\bar{2}} & \ell_{1\bar{3}} \\ \ell_{2\bar{1}} & \ell_{2\bar{2}} & \ell_{2\bar{3}} \\ \ell_{3\bar{1}} & \ell_{3\bar{2}} & \ell_{3\bar{3}} \end{bmatrix} \{\underline{\bar{e}}_\alpha\} \equiv \underline{\underline{T}}^t\{\underline{\bar{e}}_\alpha\}$$

Now, let us consider the ***Moment of Momentum*** vector about the point "O", such that:

$$\{\underline{H}^O\}_\alpha = [\underline{\underline{T}}]\{\underline{H}^O\}_i \qquad \textbf{(7.24)}$$

But:

$$\begin{cases} \{\underline{H}^O\}_\alpha = [I_\alpha^O]\{\underline{\omega}\}_\alpha \\ \{\underline{H}^O\}_i = [I_{ij}^O]\{\underline{\omega}\}_i \end{cases}$$

substituting in equation (7.24) results:

$$[I_\alpha^O]\{\underline{\omega}\}_\alpha = [\underline{\underline{T}}][I_{ij}^O]\{\underline{\omega}\}_i$$

$$= \{[\underline{\underline{T}}][I_{ij}^O][\underline{\underline{T}}^t]\}\{[\underline{\underline{T}}]\{\underline{\omega}\}_i\}$$

$$= \{[\underline{\underline{T}}][I_{ij}^O][\underline{\underline{T}}^t]\}\{\underline{\omega}\}_\alpha$$

$$[I_{\alpha}^{O}]\{\underline{\omega}\}_{\alpha} = [\underline{\underline{T}}][I_{ij}^{O}]\{\underline{\omega}\}_{i}$$

$$= \{[\underline{\underline{T}}][I_{ij}^{O}][\underline{\underline{T}}^{t}]\}\{[\underline{\underline{T}}]\{\underline{\omega}\}_{i}\}$$

$$= \{[\underline{\underline{T}}][I_{ij}^{O}][\underline{\underline{T}}^{t}]\}\{\underline{\omega}\}_{\alpha}$$

Therefore:

$$[I_{\alpha}^{O}] = [\underline{\underline{T}}][I_{ij}^{O}][\underline{\underline{T}}^{t}] \qquad \textbf{(7.25)}$$

Equation (7.25) is known as the *Rotation Transformation of Inertia Properties*. (same as Eq. 5.54 in Ginsberg Book)

Theorem-20: **For a body with a plane of symmetry, any axis perpendicular to the plane is a principal axis of inertia at the point of intersection with that plane. (*In other words, if two coordinate axes form a plane of symmetry for a body, then all product of inertias involving the coordinate normal to that plane are zero*).**

Theorem-21: **For a body having two planes of symmetry, the line of intersection is also the principal axis for any moment center lying on the line.**

Theorem-22: **If at least two of the three coordinate planes are planes of symmetry for a body, then all products of inertias are zero.**

متشکرم

For a Matrix $[\underline{\underline{A}}]$**:**

Eigenvalues: $\left|\underline{\underline{A}} - \lambda \underline{\underline{I}}\right| = 0$, ($\underline{\underline{I}}$ = ***Unit Matrix*** = δ_{ij})

Eigenvectors:

$$\left|\underline{\underline{A}} - \lambda \underline{\underline{I}}\right| \{\underline{x}\} = \{\underline{0}\}$$

, ($\underline{\underline{I}}$ = ***Unit Matrix*** = δ_{ij})

(READ EXAMPLES 5.1 and 5.3 of Ginsburg)