بسم الله الرحمن الرحیم

By: Professor Ali Meghdari

The Kinetic Principles in the Newtonian Reference Frame (NRF) :

The kinetic state of a material system is conserved unless disturbed by exterior actions. The *Principle of Momentum* (P.M.) and the *Principle of Moment of Momentum* (P.M.M) govern the change.

Recall: Newtonian (Inertial) Reference Frame;
***Non-accelerating* & *Irrotational* reference frame.**

***Admissible Newtonian Forces* in the NRF shown on a *Free-Body-Diagram* are:**

- **Contact Forces**
- **Field Forces (i.e. gravitational field, and electromagnetic field)**
- **Spring, and Friction Forces**

Momentum Principle (M.P.) for a Single Particle: **The time rate of change in the momentum of a particle is equal to the resultant external force acting on the particle.**

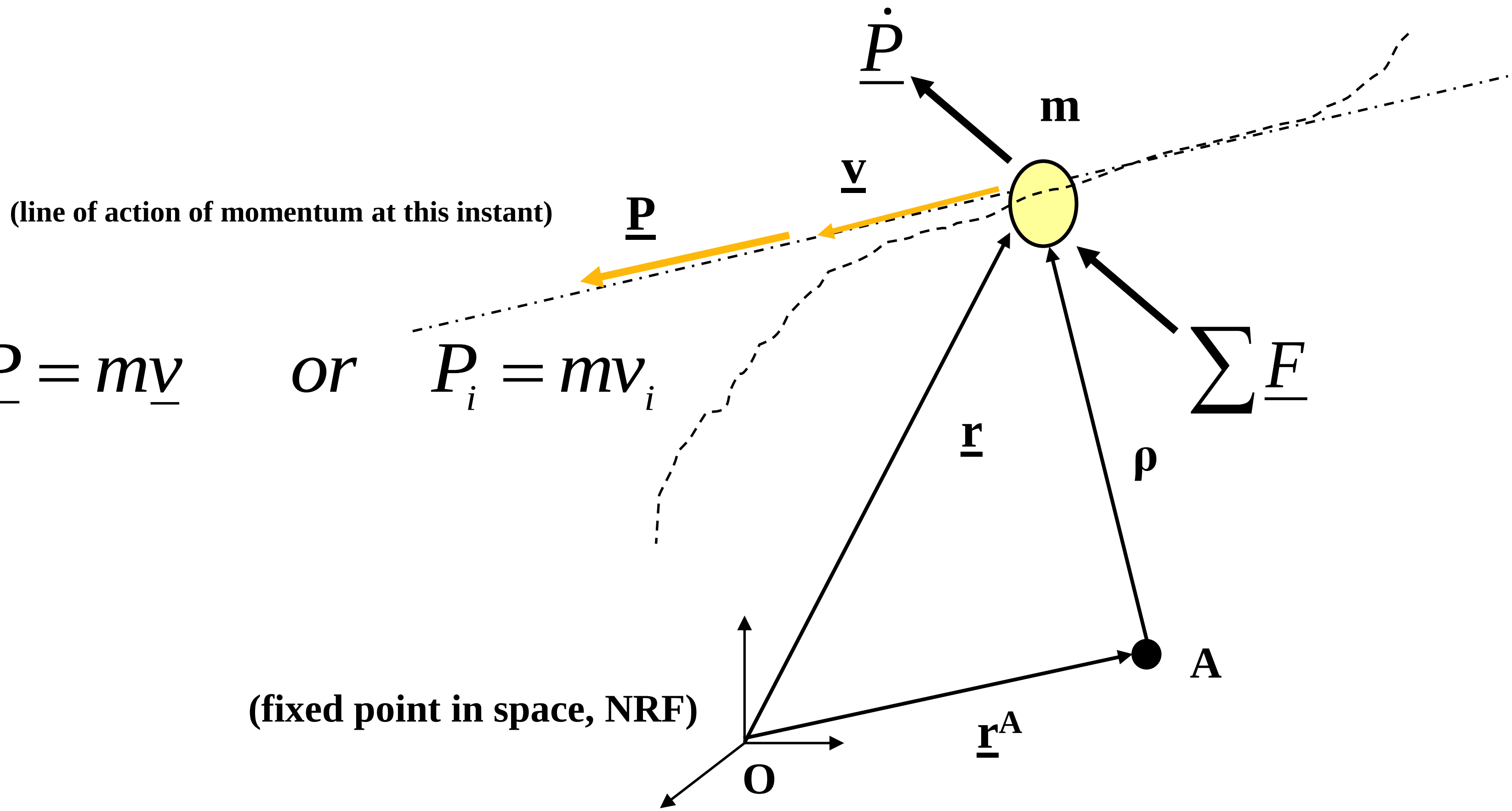
$$\underline{F} = m\underline{a} = \frac{d}{dt}(m\underline{v}) = \underline{\dot{P}} \quad \Rightarrow \quad \underline{F} = \underline{\dot{P}} \qquad \textbf{(6.20)}$$

Momentum Principle (M.P.) for a System of Particles:

$$\underline{F} = \sum_{\beta=1}^{N} \underline{\dot{P}}^{\beta} = \underline{\dot{P}}$$ **; where** $\underline{P}$ **= (*Global Momentum*)**

Moment of Momentum Principle (M.M.P.) for a Single Particle:

The time rate of change in moment of momentum of a particle about a fixed moment center "O" in space isequal to the resultant external moment (of forces) aboutthe same moment center.



$$\underline{P} = m\underline{v} \qquad or \qquad P_i = mv_i$$

$$\underline{M}^O = \underline{r} \times \underline{F} = \underline{r} \times \dot{\underline{P}}$$

$$\underline{H}^O = \underline{r} \times \underline{P} = \underline{r} \times m\underline{v}$$

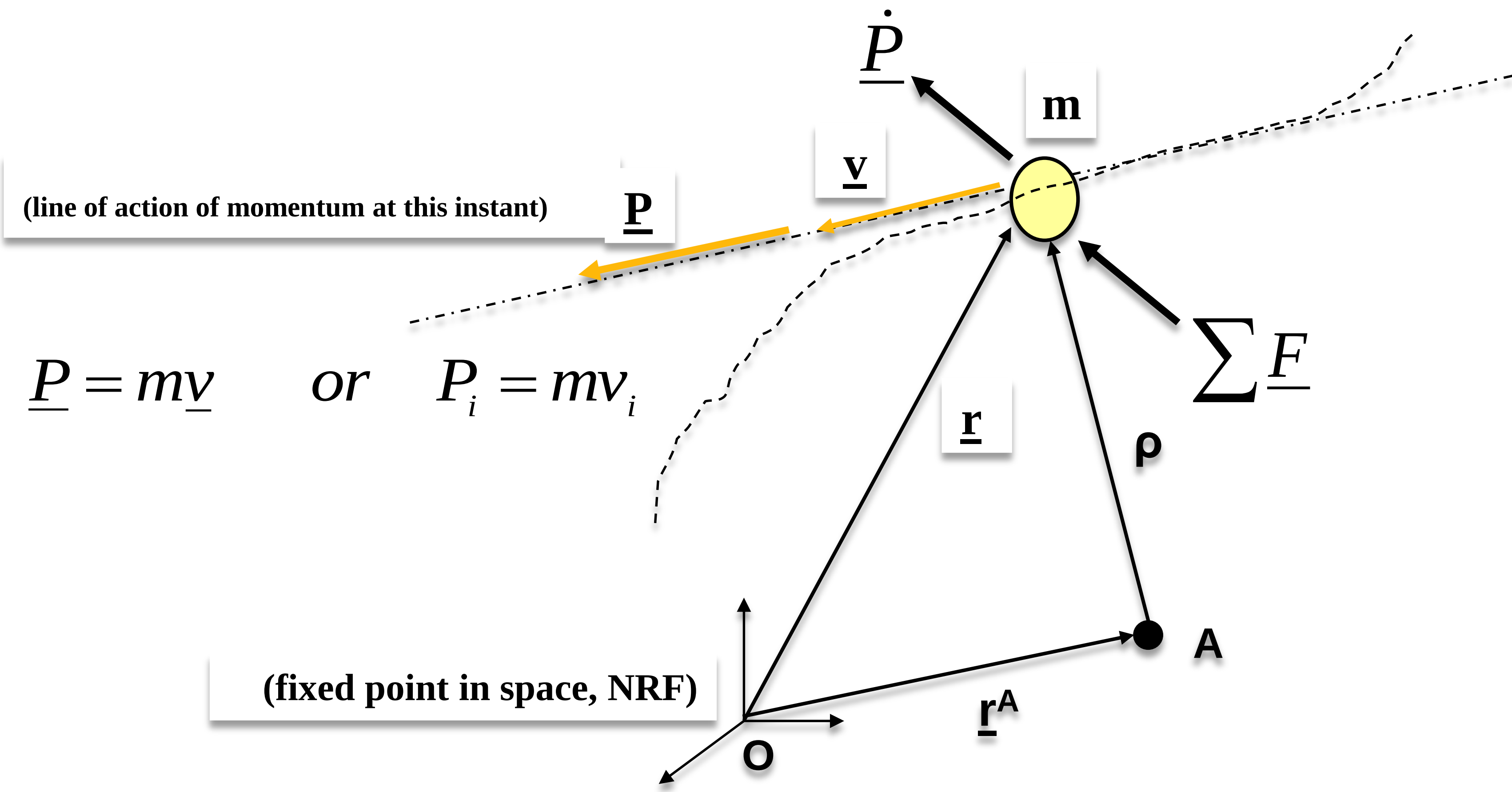
$$\dot{\underline{H}}^O = \frac{d}{dt}(\underline{r} \times \underline{P}) = \dot{\underline{r}} \times \underline{P} + \underline{r} \times \dot{\underline{P}} = \underbrace{\cancel{\dot{\underline{r}} \times m\dot{\underline{r}}}}_{0} + \underline{M}^O$$

$$\underline{M}^O = \dot{\underline{H}}^O \qquad (6.21)$$

Remark: The Moment of Momentum Principle can actually be expressed more generally for any moment center "A" which has a zero velocity in space.

$\underline{M}^A = \dot{\underline{H}}^A$ **; A= moment center with zero velocity, but may be accelerating.**

$$\underline{P} = m\underline{v} \quad or \quad P_i = m v_i$$



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***Proof*:**

$$\underline{H}^A = \underline{\rho} \times \underline{P} = (\underline{r} - \underline{r}^A) \times \underline{P}$$

$$\underline{\dot{H}}^A = \underline{\dot{r}} \times \underline{P} - \underline{\dot{r}}^A \times \underline{P} + (\underline{r} - \underline{r}^A) \times \underline{\dot{P}} =$$

$$= -\underline{v}^A \times \underline{P} + \underline{\rho} \times \underline{F} =$$

$$= -\underline{v}^A \times \underline{P} + \underline{M}^A$$

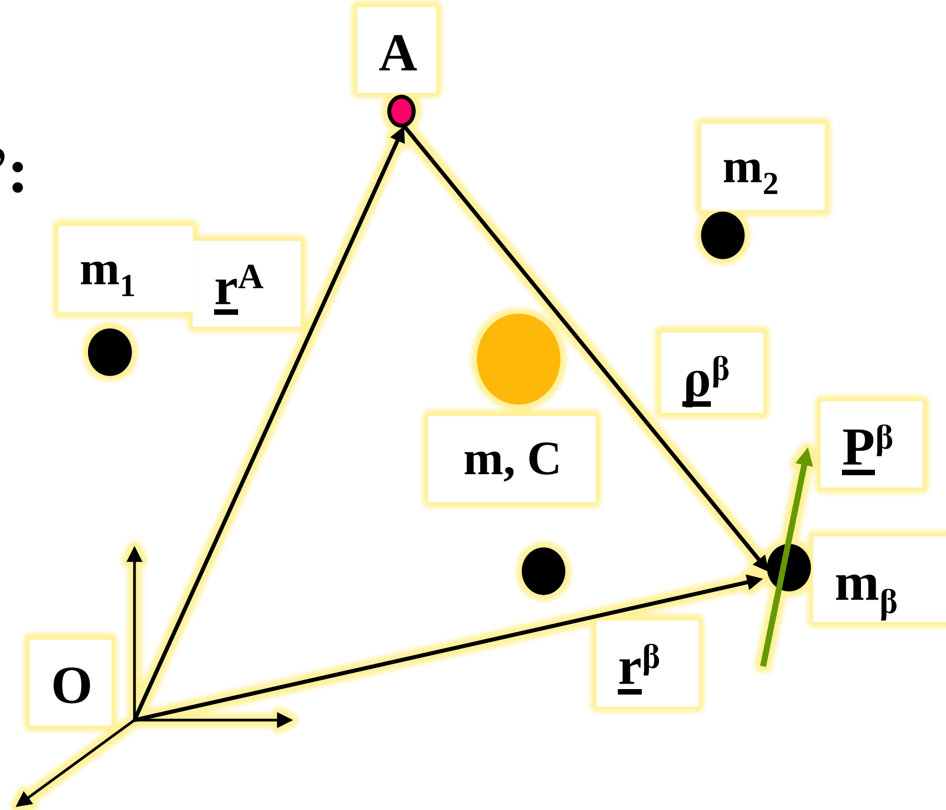
$$\underline{\dot{H}}^A = \underline{M}^A - \underline{v}^A \times \underline{P} \qquad \textbf{(6.22)}$$

If $\underline{v}^A = 0, \quad or \quad \underline{v}^A \,||\, \underline{P} \quad \Rightarrow \quad \underline{\dot{H}}^A = \underline{M}^A$

(A is an admissible moment center)

Moment of Momentum Principle (M.M.P.) for a System of Particles:

- **About a *fixed point* "O":**



$$\underline{M}^{O} = \dot{\underline{H}}^{O}$$

$$where: \begin{cases} \underline{H}^{O} = \sum_{\beta=1}^{N} \underline{r}^{\beta} \times \underline{P}^{\beta}, \\ and: \underline{M}^{O}: \text{Resultant Moment} \end{cases}$$

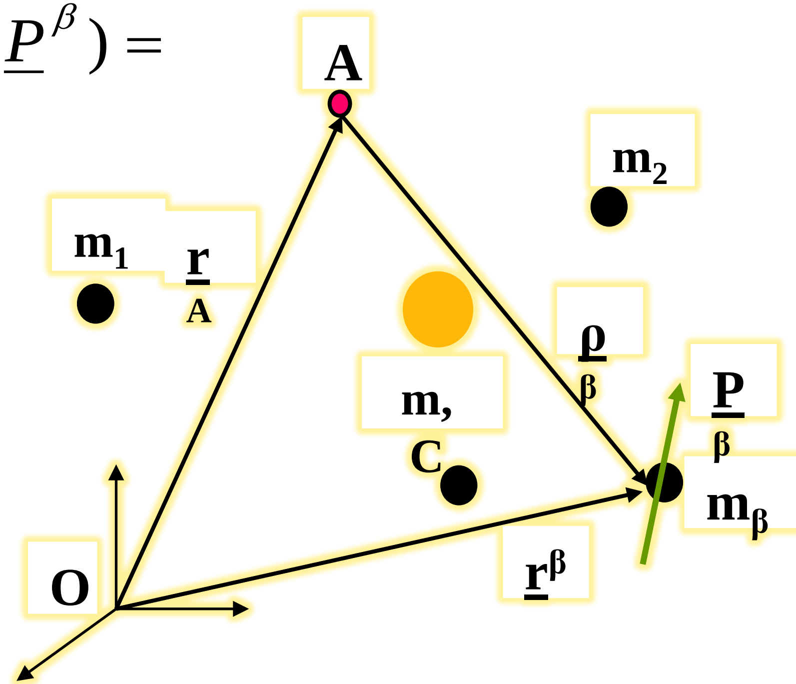
- About a ***general point*** "A":

$$\dot{\underline{H}}^{A} = \frac{d}{dt}\sum_{\beta=1}^{N}\underline{\rho}^{\beta}\times\underline{P}^{\beta} = \sum_{\beta=1}^{N}\underline{\rho}^{\beta}\times\dot{\underline{P}}^{\beta} + \sum_{\beta=1}^{N}\dot{\underline{\rho}}^{\beta}\times\underline{P}^{\beta} =$$

$$= \underline{M}^{A} + \Big(\underbrace{\sum_{\beta=1}^{N}\dot{\underline{r}}^{\beta}\times\underline{P}^{\beta}}_{0} - \sum_{\beta=1}^{N}\dot{\underline{r}}^{A}\times\underline{P}^{\beta}\Big) =$$

$$= \underline{M}^{A} - \Big(\underline{v}^{A}\times\sum_{\beta=1}^{N}\underline{P}^{\beta}\Big)$$

$$\dot{\underline{H}}^{A} = \underline{M}^{A} - (\underline{v}^{A}\times\underline{P})$$



Since: $\underline{\rho}^{\beta} = \underline{r}^{\beta} - \underline{r}^{A} \Rightarrow \dot{\underline{\rho}}^{\beta} = \dot{\underline{r}}^{\beta} - \dot{\underline{r}}^{A}$

$$\underline{\dot{H}}^{A} = \underline{M}^{A} - (\underline{v}^{A} \times \underline{P}) \qquad \textbf{(6.23)}$$

However, if

$$\left\{ \begin{array}{l} \underline{v}^{A} = 0 \\ A \equiv C \\ \underline{v}^{A} \,\|\, \underline{P} \end{array} \right\} \Rightarrow \underline{\dot{H}}^{A} = \underline{M}^{A} \qquad \textbf{(6.24)}$$

Since for a constant mass system: $\underline{P} = m\underline{v}^{C}$. **Point "A" is an *admissible moment center* for equation (6.24), if and only if it satisfies one of the above conditions.**

Linear and Angular Impulse:

- ***Linear Impulse of a Force*** **"F": Let; F=F(t) and P=mv , then:**

$$\underline{F} = \dot{\underline{P}} = \frac{d}{dt}\underline{P} \Rightarrow \underline{F}dt = d\,\underline{P} \Rightarrow$$

$$\underline{f} = \underline{impulsive - force} = \int_{t_1}^{t_2} \underline{F}dt = \int_{t_1}^{t_2} d\,\underline{P} = \underline{P}(t_2) - \underline{P}(t_1) \qquad \textbf{(6.25)}$$

- ***Angular Impulse of a Moment*** **"M": Let;** $\underline{M}^O = \underline{M}^O(t)$ **and** $\underline{H}^O = \underline{r} \times \underline{P}$ **, then:**

$$\underline{M}^O = \dot{\underline{H}}^O = \frac{d}{dt}\underline{H}^O \Rightarrow \underline{M}^O dt = d\,\underline{H}^O \Rightarrow \qquad \textbf{(6.26)}$$

$$\underline{M} = \underline{impulsive - moment} = \int_{t_1}^{t_2} \underline{M}^O dt = \int_{t_1}^{t_2} d\,\underline{H}^O = \underline{H}^O(t_2) - \underline{H}^O(t_1)$$

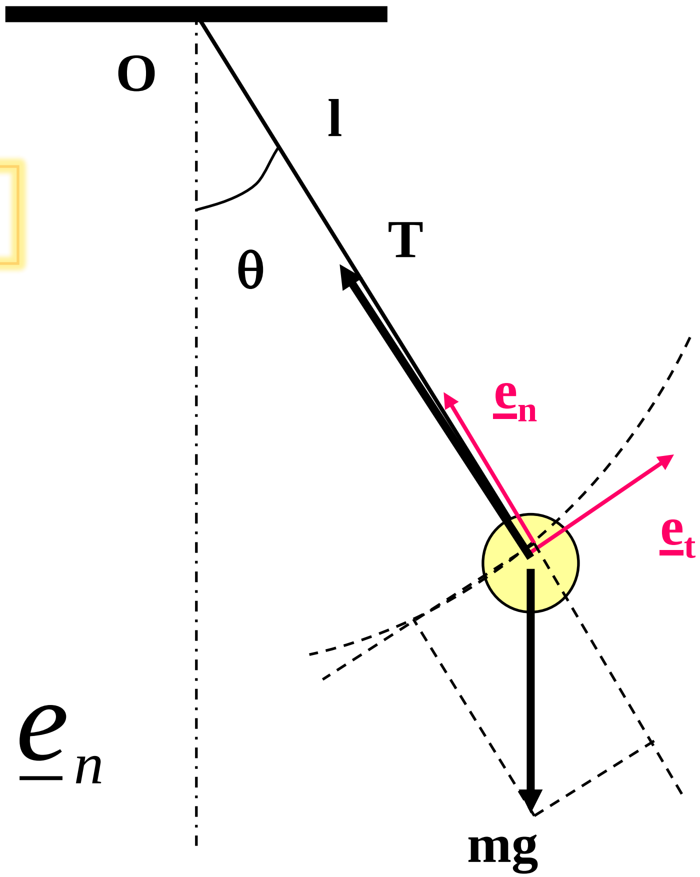
Note: ***the "Momentum" and the "Moment of Momentum" are conserved if the Impulse Integral Vanishes*****, that is when:**

1. $\underline{F} = \underline{M}^{O} = 0$ **(the integrand vanishes).**

2. **The integrand is finite, but the time interval is infinitesimal ($\Delta t = \varepsilon$), (i.e. impact problems, and explosions).**

3. **The integrand is a cyclic function and the time interval is one complete period, (i.e. materials under cyclic loading).**

Ex: **A simple pendulum constrained to oscillate in a plane as shown, has a mass "m" and a length "L". Derive its differential equation of motion?**



***Solution*:**

(a). By the *Momentum Principle*:

- **Motion Analysis:**

$$\underline{v} = \ell \dot{\theta} \underline{e}_t$$

$$\underline{a} = \ell \ddot{\theta} \underline{e}_t + \ell \dot{\theta}^2 \underline{e}_n$$

- **Free Body Diagram**

❖ **Kinetics (M.P.):**

$$\underline{f} = \underline{\dot{P}} = m\underline{a}$$

$$T\underline{e}_n - mg(\cos\theta\underline{e}_n + \sin\theta\underline{e}_t) = m\underline{a} = m(\ell\ddot{\theta}\underline{e}_t + \ell\dot{\theta}^2\underline{e}_n)$$

$$\begin{cases} \underline{e}_t: \quad -mg\sin\theta = m\ell\ddot{\theta} \quad \Rightarrow \ddot{\theta} + \dfrac{g}{\ell}\sin\theta = 0 \\ \underline{e}_n: \quad T - mg\cos\theta = m\ell\dot{\theta}^2 \quad \Rightarrow T = m(\ell\dot{\theta}^2 + g\cos\theta) \end{cases}$$

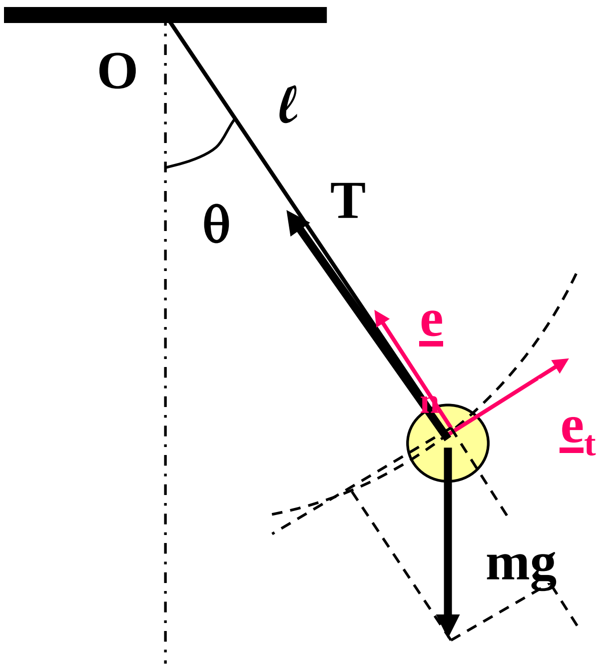
D.E.M.

(b). By the *Momentum of Momentum Principle*:

- **Motion Analysis:**

$$\underline{v} = \ell\dot{\theta}\underline{e}_t$$

- **Free Body Diagram**



- **Kinetics (M.M.P.):**

$$\begin{cases} \underline{P} = m\underline{v} = m\ell\dot{\theta}\underline{e}_t \\ \underline{H}^O = \underline{r}\times\underline{P} = (-\ell\underline{e}_n)\times(m\ell\dot{\theta}\underline{e}_t) = m\ell^2\dot{\theta}\underline{e}_b \end{cases}$$

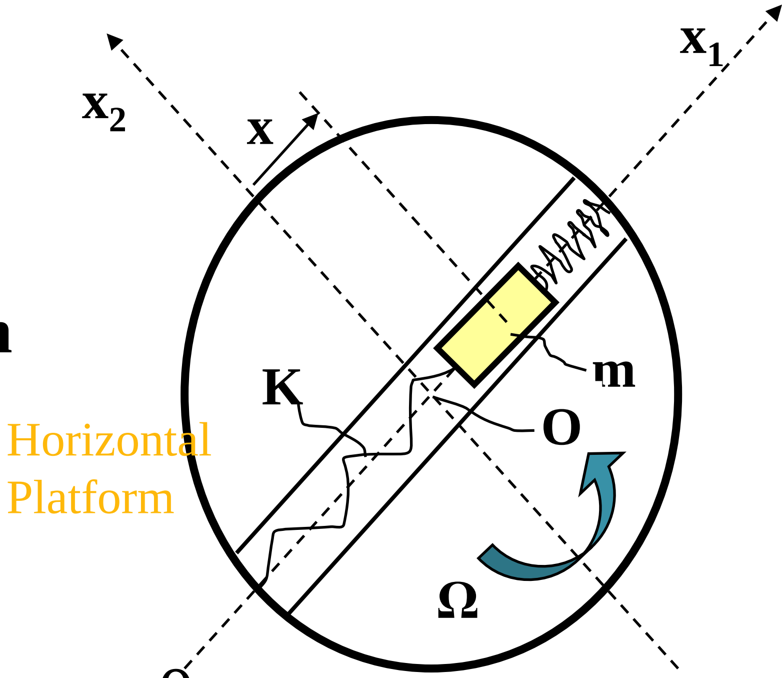
$$\begin{cases} \underline{M}^O = \underline{\dot{H}}^O \quad \Rightarrow -mg\ell\sin\theta = m\ell^2\ddot{\theta} \\ \ddot{\theta} + \dfrac{g}{\ell}\sin\theta = 0 \end{cases}$$

Ex: A horizontal table is spinning at a constant angular velocity of "Ω". On the table, a spring-retained mass "m" is oscillating in a straight slot as shown. The spring constant of each spring is "k". Determine the equation of motion of the mass, if the coefficient of friction between the mass and the slot is "μ".

Solution:

❖ **Motion Analysis:**

a). Fix KRF $\{x_i\}$ to the table with reference point at "O".



b). KRF Motion;

$$\underline{\Omega} = \Omega \underline{e}_3, \qquad \dot{\Omega} = 0, \qquad \underline{v}_O = \underline{a}_O = 0$$

c). Relative Motion;

$$\underline{\bar{r}} = x\underline{e}_1, \qquad \underline{\bar{v}} = \dot{x}\underline{e}_1, \qquad \underline{\bar{a}} = \ddot{x}\underline{e}_1$$

d). Absolute Motion;

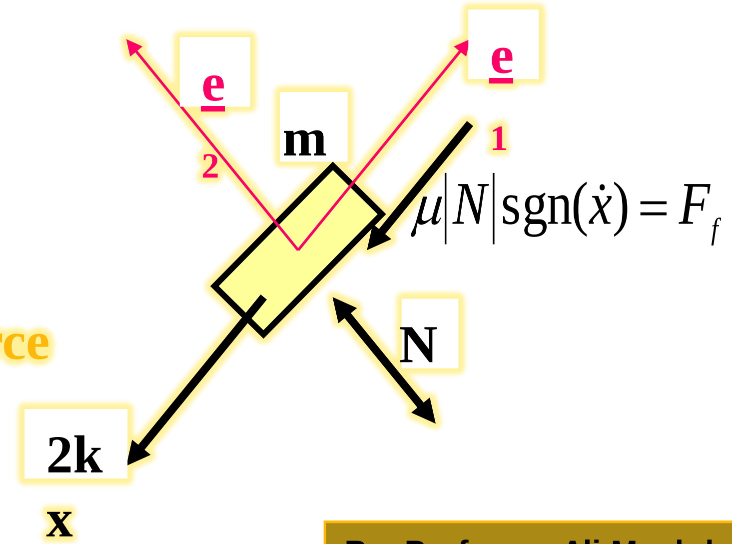
$$\underline{a}_m = \underline{a}_O + \underline{\bar{a}} + \dot{\underline{\Omega}} \times \underline{\bar{r}} + 2\underline{\Omega} \times \underline{\bar{v}} + \underline{\Omega} \times (\underline{\Omega} \times \underline{\bar{r}})$$

$$\underline{a}_m = (\ddot{x} - \Omega^2 x)\underline{e}_1 + 2\Omega\dot{x}\underline{e}_2$$

❖ **Free Body Diagram**

Coulomb (Dry) Friction:

$\mu|N|\operatorname{sgn}(\dot{x}) = F_f$ **, a constant force to oppose the sliding motion.**



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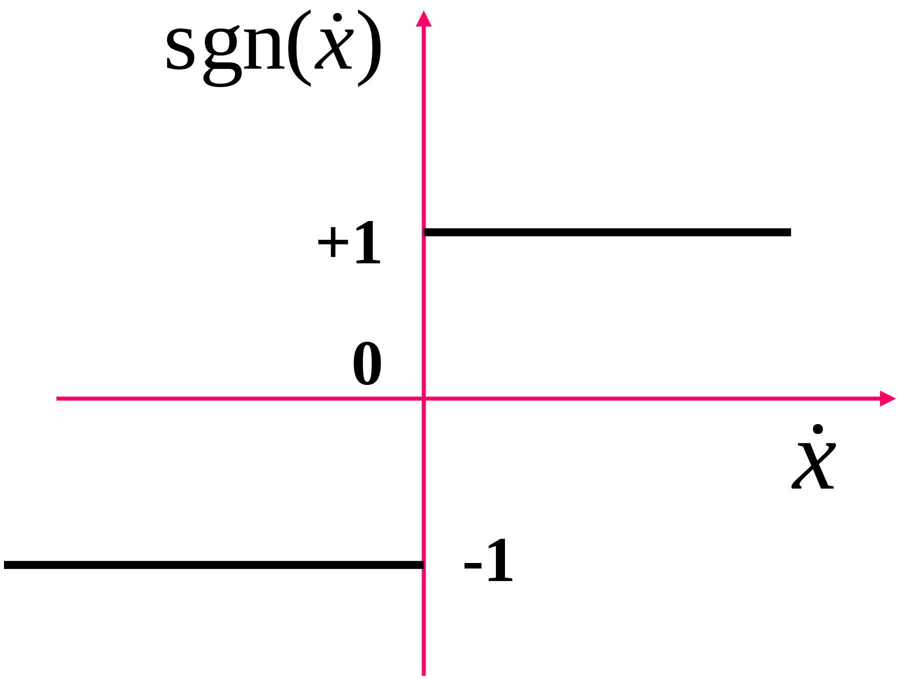
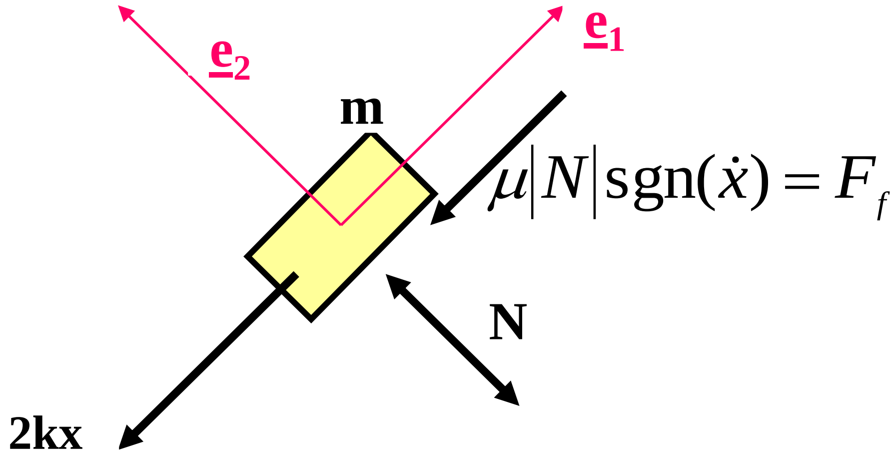
Signum Function:



$$\mathrm{sgn}(\dot{x}) = +1 \qquad if \quad \dot{x} > 0$$

$$\mathrm{sgn}(\dot{x}) = -1 \qquad if \quad \dot{x} < 0$$

$$-1 \le \mathrm{sgn}(\dot{x}) \le +1 \qquad if \quad \dot{x} = 0$$



❖ Kinetic Analysis

Momentum Principle; $\underline{f} = \underline{\dot{P}} = m\underline{a}$

$$(-2kx - \mu|N|\text{sgn}(\dot{x}))\underline{e}_1 + N\underline{e}_2 = m[(\ddot{x} - \Omega^2 x)\underline{e}_1 + 2\Omega\dot{x}\underline{e}_2]$$

$$\underline{e}_2: \qquad N = 2m\Omega\dot{x}$$

$$\underline{e}_1: \qquad -2kx - \mu|N|\text{sgn}(\dot{x}) = m(\ddot{x} - \Omega^2 x)$$

$$-2kx - 2\mu m|\Omega||\dot{x}|\text{sgn}(\dot{x}) = m(\ddot{x} - \Omega^2 x)$$

$$m\ddot{x} + 2\mu m|\Omega|\dot{x} + (2k - m\Omega^2)x = 0, \quad or$$

$$\ddot{x} + 2\mu|\Omega|\dot{x} + (\frac{2k}{m} - \Omega^2)x = 0$$

D.E.M.

(*Linearly decaying oscillation due to the dry friction*). Note that direction of "N" is arbitrary and does not affect the D.E.M.

Moment of Momentum Principle (M.M.P.) in terms of a rotating coordinate (KRF) with angular velocity "$\underline{\Omega}$":

$$\underline{M}^O = \dot{H}_i^O \underline{u}_i + \underline{\Omega} \times \underline{H}^O \qquad or \qquad M_i^O = \dot{H}_i^O + \gamma_{ijk}\Omega_j H_k^O$$

(6.27)

متشکرم