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REVIEW:

Time Derivative of a Vector: For a general vector **A = Ae$_A$**, $\boxed{\dot{\underline{A}} = \dot{A}\underline{e}_A + \underline{\omega}_A \times \underline{A}}$ {*Jaumann Rate of a Vector*}.

The time derivative in terms of its *Cartesian* component set {**A$_i$**}, where; $\underline{A} = A_i(t)\underline{e}_i(t)$, is: $\boxed{\dot{\underline{A}} = \dot{A}_i\underline{e}_i + \underline{\Omega}\times\underline{A}}$

PARTICLE KINEMATICS:

Path Variables Description (Intrinsic Coordinates): **s = s(t)**: arc length; **$\underline{r}_{P/O}$ = $\underline{r}_{P/O}$(s)** =***Position Vector** at time* "t";

Velocity: $\underline{v}_P = \dot{s}\underline{e}_t = v\underline{e}_t; \underline{e}_t$: (unit vector tangent to the path); ***Acceleration***: $\underline{a} = \dot{v}\underline{e}_t + \frac{v^2}{\rho}\underline{e}_n = a_t\underline{e}_t + a_n\underline{e}_n$

$\underline{e}_n$ ⊥ **$\underline{e}_t$** , and normal to the path directing toward the center of curvature. For a *planar curve or path* like "**y=f(x)**, the Radius of Curvature "**ρ**" is computed from:

$$\boxed{\frac{1}{\rho} = \frac{\left|\frac{d^2y}{dx^2}\right|}{[1+(\frac{dy}{dx})^2]^{3/2}} = \frac{|\ddot{y}|}{[1+(\dot{y})^2]^{3/2}}}$$ and, $$\boxed{s = \int_{x_0}^{x}[1+(\frac{dy}{dx})^2]^{1/2}dx}$$ = *The Arc Length*

For a particle traveling on a path or *a curve in 3-dimension* (x,y,z) coordinate so that its path is described by the *position vector* "**$\underline{r}$**" as a function of the parameter "**t** *within a possible range*", we have: $\underline{r} = x(t)\underline{e}_1 + y(t)\underline{e}_2 + z(t)\underline{e}_3$, then:

$$\boxed{\frac{1}{\rho} = \frac{1}{(\dot{s})^3}[(\ddot{\underline{r}}\cdot\ddot{\underline{r}})(\dot{s})^2 - (\dot{\underline{r}}\cdot\ddot{\underline{r}})^2]^{1/2}}$$ where: $\dot{s} = (\dot{\underline{r}}\cdot\dot{\underline{r}})^{1/2} = [(\dot{x})^2 + (\dot{y})^2 + (\dot{z})^2]^{1/2}$,and

$$\boxed{s = \int_{t_0}^{t}[(\dot{x})^2 + (\dot{y})^2 + (\dot{z})^2]^{1/2}dt \equiv (\underline{Arc - Length})}$$

$$\boxed{\begin{aligned} \underline{e}_t &= \frac{d\underline{r}}{ds} = \frac{d\underline{r}}{dt}\frac{dt}{ds} = \frac{\dot{\underline{r}}}{\dot{s}} \\ \underline{e}_n &= \rho\frac{d\underline{e}_t}{ds} = \rho\frac{d\underline{e}_t}{dt}\frac{dt}{ds} = \frac{\rho}{(\dot{s})^4}[\ddot{\underline{r}}(\dot{s})^2 - \dot{\underline{r}}(\dot{\underline{r}}\cdot\ddot{\underline{r}})] \\ \underline{e}_b &= \underline{e}_t \times \underline{e}_n = \frac{\rho}{(\dot{s})^3}\dot{\underline{r}}\times\ddot{\underline{r}} \equiv (\underline{binormal - unit - vector}) \end{aligned}}$$

Cartesian (Rectangular) Coordinates {x$_i$}: $\underline{r} = x_1\underline{e}_1 + x_2\underline{e}_2 + x_3\underline{e}_3 = x_i\underline{e}_i$ = (*Position Vector*);

$\underline{v} = \dot{\underline{r}} \equiv v_i\underline{e}_i = \dot{x}_i\underline{e}_i$ = (*Particle Velocity*); $\underline{a} = \dot{\underline{v}} = \ddot{\underline{r}} \equiv a_i\underline{e}_i = \ddot{x}_i\underline{e}_i$ = (*Particle Acceleration*).

Matrix of Direction Cosines between coordinate { $\bar{x}_j$ } and { x_i }:

$$\underline{\underline{T}} = [\ell_{ij}] = \begin{bmatrix} l_{1\bar{1}} & l_{1\bar{2}} & l_{1\bar{3}} \\ l_{2\bar{1}} & l_{2\bar{2}} & l_{2\bar{3}} \\ l_{3\bar{1}} & l_{3\bar{2}} & l_{3\bar{3}} \end{bmatrix}$$; where: $\ell_{1\bar{1}} = \underline{e}_1\cdot\bar{\underline{e}}_1 = |\underline{e}_1||\bar{\underline{e}}_1|\cos(\underline{e}_1,\bar{\underline{e}}_1) = \cos(x_1,\bar{x}_1)$

Columns of $\underline{\underline{T}} = [\ell_{ij}]$ are the projection of the unit vectors of $\bar{x}_j$ into x_i. ***Rows*** of $\underline{\underline{T}} = [\ell_{ij}]$ are the projection of the unit vectors of x_i into $\bar{x}_j$. Therefore: $\{\underline{P}\}_x = \underline{\underline{T}}\{\underline{P}\}_{\bar{x}}$ *or* $\{\underline{x}\} = \underline{\underline{T}}\{\bar{\underline{x}}\}$.

When the two origins coincide, $\underline{\underline{T}}$ is a ***Rotation Matrix***, and ***Orthogonal*** **(** $\underline{\underline{T}}^{-1} = \underline{\underline{T}}^t$ **)**, expressing the relative orientation of frame { $\bar{x}_j$ } with respect to { x_i }.

Elementary Rotation Matrices: Rotations about one of the coordinate axes.

$$\underline{\underline{R}}_1(x_1,\theta_1)=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\theta_1 & -\sin\theta_1\\ 0 & \sin\theta_1 & \cos\theta_1\end{bmatrix};\ \underline{\underline{R}}_2(x_2,\theta_2)=\begin{bmatrix}\cos\theta_2 & 0 & \sin\theta_2\\ 0 & 1 & 0\\ -\sin\theta_2 & 0 & \cos\theta_2\end{bmatrix};$$

$$\underline{\underline{R}}_3(x_3,\theta_3)=\begin{bmatrix}\cos\theta_3 & -\sin\theta_3 & 0\\ \sin\theta_3 & \cos\theta_3 & 0\\ 0 & 0 & 1\end{bmatrix}$$

ORTHOGONAL CURVILINEAR COORDINATES:

Cylindrical Coordinates (R, φ, Z), and $\{\underline{x}_R, \underline{x}_\varphi, \underline{x}_Z\}$: *Base Vectors*

Position Vector: $\boxed{\underline{r}_{P/O}=R\underline{e}_R+Z\underline{e}_Z}$; ***Velocity Vector***: $\boxed{\underline{v}_P=\dot{R}\underline{e}_R+R\dot{\varphi}\underline{e}_\varphi+\dot{Z}\underline{e}_Z}$;

Acceleration Vector: $\boxed{\underline{a}_P=(\ddot{R}-R\dot{\varphi}^2)\underline{e}_R+(2\dot{R}\dot{\varphi}+R\ddot{\varphi})\underline{e}_\varphi+\ddot{Z}\underline{e}_Z=a_R\underline{e}_R+a_\varphi\underline{e}_\varphi+a_Z\underline{e}_Z}$

Spherical Coordinates (θ, φ, R), and $\{\underline{x}_\theta, \underline{x}_\varphi, \underline{x}_R\}$: *Base Vectors*

Position Vector: $\boxed{\underline{r}_{P/O}=R\underline{e}_R}$; ***Velocity Vector***: $\boxed{\underline{v}_P=\dot{\underline{r}}_{P/O}=\dot{R}\underline{e}_R+(R\dot{\varphi}\sin\theta\underline{e}_\varphi+R\dot{\theta}\underline{e}_\theta)}$;

Acceleration Vector:

$$\boxed{\begin{aligned}\underline{a}_P=&(\ddot{R}-R\dot{\varphi}^2\sin^2\theta-R\dot{\theta}^2)\underline{e}_R+(2\dot{R}\dot{\varphi}\sin\theta+2R\dot{\theta}\dot{\varphi}\cos\theta+R\ddot{\varphi}\sin\theta)\underline{e}_\varphi+\\&+(R\ddot{\theta}+2\dot{R}\dot{\theta}-R\dot{\varphi}^2\sin\theta\cos\theta)\underline{e}_\theta\end{aligned}}$$

Theorem-6: The orientation of a curvilinear coordinate, $\mathbf{q}^\alpha$**,** (i.e. R, φ, Z), at a point "P" in space is defined by the direction of the ***Base Vectors***, $\underline{g}_\alpha$, at that point. If $\underline{r}(q^\alpha)$=position vector of "P", then:

Base Vectors = $\underline{g}_\alpha=\dfrac{\partial \underline{r}}{\partial q^\alpha}$, and ***unit vectors*** of $\underline{g}_\alpha$ are: $\underline{e}_\alpha=\dfrac{\underline{g}_\alpha}{|\underline{g}_\alpha|}$. *Note* that the curvilinear coordinates are *Orthogonal* if the base vectors form an orthogonal set, that is: "$\underline{g}_\alpha\cdot\underline{g}_\beta$ = 0 for α ≠ β ".

RIGID BODY KINEMATICS:

Simple Rotation: Rotation of a rigid body about a general fixed axes in space.

Euler's Theorem: Any change of orientation (about an arbitrary axis) for a rigid body with a fixed body point can be accomplished through a *simple rotation* $\underline{\underline{R}}$. Then, the rigid body rotation can be resolved into three *elementary rotations*, where the angles of these rotations are called the ***Euler's Angles***.

Two situations commonly arise in sequential rotations:

1. *Body Fixed Rotations* (*Rotations about New-axes*): For n # of rotations: $\underline{\underline{R}}=\underline{\underline{R}}_1\underline{\underline{R}}_2\underline{\underline{R}}_3\ldots\underline{\underline{R}}_n$ =(***Post-multiply***) →

2. *Space Fixed Rotations* (*Rotations about Old-axes*)**:** for n # of rotations: $\underline{\underline{R}}=\underline{\underline{R}}_n\ldots\underline{\underline{R}}_3\underline{\underline{R}}_2\underline{\underline{R}}_1$ =(***Pre-multiply***) ←

Rotation About an Arbitrary Axis (Equivalent Angle-Axis Representation):
Euler's Theorem(continued)*:** Any change of orientation for a rigid body with a fixed body point can be accomplished through a ***General Rotation Operator (a simple rotation) with a proper axis and angle selection, where:

$$\boxed{\underline{\underline{R}}({}^x\underline{K},\theta)=\begin{bmatrix}k_{x1}k_{x1}v\theta+c\theta & k_{x1}k_{x2}v\theta-k_{x3}s\theta & k_{x1}k_{x3}v\theta+k_{x2}s\theta\\ k_{x1}k_{x2}v\theta+k_{x3}s\theta & k_{x2}k_{x2}v\theta+c\theta & k_{x2}k_{x3}v\theta-k_{x1}s\theta\\ k_{x1}k_{x3}v\theta-k_{x2}s\theta & k_{x2}k_{x3}v\theta+k_{x1}s\theta & k_{x3}k_{x3}v\theta+c\theta\end{bmatrix}}$$

$$ {}^{x}\underline{K} = k_{x1}\underline{e}_1 + k_{x2}\underline{e}_2 + k_{x3}\underline{e}_3 \quad and \quad k_{x1}^2 + k_{x2}^2 + k_{x3}^2 = 1 $$, and $v\theta = vers\theta = (1-\cos\theta)$.

For a given Rotation Matrix like $\underline{\underline{R}} = {}^{x}_{\xi}\underline{\underline{R}} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$, one can determine *the equivalent angle-axis* by taking an inverse approach, by setting $\underline{\underline{R}} = {}^{x}_{\xi}\underline{\underline{R}}({}^{x}\underline{K}, \theta)$, and solving to obtain:

$$\sin\theta = \pm\frac{1}{2}\sqrt{(r_{32}-r_{23})^2 + (r_{13}-r_{31})^2 + (r_{21}-r_{12})^2}$$, and $$\cos\theta = \frac{r_{11}+r_{22}+r_{33}-1}{2}$$, where:

$$\theta = \tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right); \text{ and } {}^{x}\underline{K} = \frac{1}{2\sin\theta}\begin{bmatrix} r_{32}-r_{23} \\ r_{13}-r_{31} \\ r_{21}-r_{12} \end{bmatrix} = \begin{bmatrix} k_{x1} \\ k_{x2} \\ k_{x3} \end{bmatrix}$$

This solution is valid for ($0\langle\theta\langle 180$), and for every pair of equivalent angle-axis $({}^{x}\underline{K}, \theta)$, there exists another pair as $(-{}^{x}\underline{K}, -\theta)$ representing the same orientation in space with the same rotation matrix. (no solutions for θ=0 and 180).

☺ ***<u>Any combination of Rotations is always equivalent to a single rotation about some axis "K" by an angle "θ".</u>***

For ***<u>General Infinitesimal Rotations</u>*** we have:

$$\underline{\underline{R}} = \underline{\underline{R}}_1\underline{\underline{R}}_2\underline{\underline{R}}_3 = \underline{\underline{R}}_3\underline{\underline{R}}_2\underline{\underline{R}}_1 \equiv \begin{bmatrix} 1 & -\Delta\theta_3 & \Delta\theta_2 \\ \Delta\theta_3 & 1 & -\Delta\theta_1 \\ -\Delta\theta_2 & \Delta\theta_1 & 1 \end{bmatrix}, or \quad R_{jk} = \delta_{jk} - \gamma_{ijk}\Delta\theta_i$$

<u>Angular Velocity Vector for a R.B.</u>: $\underline{\omega} = \omega_i\underline{e}_i$; If the angular velocity "$\underline{\omega}$" is defined in a set of *<u>moving coordinate</u>* {z_i} having an angular velocity "$\underline{\Omega}$", we may apply the *<u>Jaumann</u>* rate to compute the angular acceleration vector as:

<u>Angular Acceleration Vector for a R.B.</u>: $\boxed{\underline{\alpha} = \dot{\omega}_i\underline{e}_i + \underline{\Omega}\times\underline{\omega}}$.

<u>Velocity and Acceleration Field in a Rotating (only) Rigid Body</u>: ρ : A constant magnitude vector fixed in the R.B.

<u>Velocity and Acceleration</u> of a point **P** in the R.B.: $\boxed{\underline{v}_P = \dot{\underline{\rho}} = \underline{\omega}\times\underline{\rho}}$; $\boxed{\underline{a}_P = \underline{\alpha}\times\underline{\rho} + \underline{\omega}\times(\underline{\omega}\times\underline{\rho})}$.

<u>General Motion of a Rigid Body (Translation & Rotation)</u>:
<u>Chasle's Theorem</u>*:** The general motion of a rigid body can be described by a combination of motion of some convenient reference body point and an ***<u>Eulerian</u> rotation about that point.
<u>Note</u>: A rigid body in space possesses ***<u>Six-Degrees-of-Freedom</u>*** (3-DOF: for the *<u>position</u>* of the reference point on the rigid body, and 3-DOF: for the *<u>orientation</u>* of the rigid body (i.e. Euler's Angles).

<u>For a Moving Rigid body in Space</u> with some known $\underline{\omega} = \omega_i\underline{e}_i$, $\underline{\alpha} = \alpha_i\underline{e}_i$, and motion of a body point "**O**", we can compute the motion of another body point "**P**" using: ***<u>Position Vector</u>***: $\underline{r}_{P/O'} = \underline{r}_{O/O'} + \underline{\rho}$;

<u>Velocity Vector</u>: $\boxed{\underline{v}_P = \underline{v}_O + \underline{\omega}\times\underline{\rho}}$; ***<u>Acceleration Vector</u>***: $\boxed{\underline{a}_P = \underline{a}_O + \underline{\alpha}\times\underline{\rho} + \underline{\omega}\times(\underline{\omega}\times\underline{\rho})}$.

<u>KINEMATICS (MOVING) REFERENCE FRAME (KRF/MRF)</u>:

If {A}=$\{x_i\}$: ***<u>Absolute Reference Frame</u>***, and {B}= $\{\bar{x}_j\}$: ***<u>Moving (Kinematics/Rigid Body) Reference Frame</u>***,
Then motion of a point or particle in these frames may be described by:

<u>Particle's Absolute Position</u>: $\boxed{\underline{r}_{P/O} = \underline{r}_{O'/O} + \underline{\bar{r}}_{P/O'}}$

<u>Particle's Absolute Velocity</u>: $\boxed{\underline{v}_P = \underline{v}_{O'} + \underline{\bar{v}}_{P/O'} + \underline{\Omega}\times\underline{\bar{r}}_{P/O'}}$

<u>Particle's Absolute Acceleration</u>: $\boxed{\underline{a}_P = \underline{a}_{O'} + \underline{\bar{a}}_{P/O'} + \dot{\underline{\Omega}}\times\underline{\bar{r}}_{P/O'} + 2\underline{\Omega}\times\underline{\bar{v}}_{P/O'} + \underline{\Omega}\times(\underline{\Omega}\times\underline{\bar{r}}_{P/O'})}$

Special Cases :

1. If **KRF** is *Translatory/Irrotational* Reference Frame $(\underline{\Omega}=\dot{\underline{\Omega}}=0)$, then: $\begin{cases} \underline{v}_P = \underline{v}_{O'} + \overline{\underline{v}}_{P/O'} \\ \underline{a}_P = \underline{a}_{O'} + \overline{\underline{a}}_{P/O'} \end{cases}$

2. If the point **P** is a *Fixed* point in the **KRF** $(\overline{\underline{v}}_{P/O'} = \overline{\underline{a}}_{P/O'} = 0, \quad \underline{\Omega} = \underline{\omega}_{R.B.} = \underline{\omega}, \quad \dot{\underline{\Omega}} = \underline{\alpha}_{R.B.} = \underline{\alpha})$, then:

$$\underline{v}_P = \underline{v}_{O'} + \underline{\omega} \times \overline{\underline{r}}_{P/O'}$$

$$\underline{a}_P = \underline{a}_{O'} + + \underline{\alpha} \times \overline{\underline{r}}_{P/O'} + \underline{\omega} \times (\underline{\omega} \times \overline{\underline{r}}_{P/O'})$$

Note that to study the *relative motion* of a particle **P** as observed in the **KRF**, we need the **Relative Path**: $\mathbf{\Gamma_{rel.}}$.

***Rigid Body Motion (Using KRF)*:**

Theorem-14: The **Angular Velocity** of a rigid body "$\underline{\omega}$" is the vector sum of the angular velocity of the KRF "$\underline{\Omega}$" and the relative angular velocity of the rigid body in the KRF "$\overline{\underline{\omega}}$", that is: $\boxed{\underline{\omega} = \underline{\Omega} + \overline{\underline{\omega}}}$.

Theorem-15: The **Angular Acceleration** of a rigid body "$\underline{\alpha}$" is related to the coordinate angular acceleration "$\dot{\underline{\Omega}}$", the relative angular acceleration $\overline{\underline{\alpha}}$, as well as the angular velocity properties, that is: $\boxed{\underline{\alpha} = \dot{\underline{\Omega}} + \overline{\underline{\alpha}} + \underline{\Omega} \times \overline{\underline{\omega}}}$.

NEWTONIAN MECHANICS: Newtonian reference Frame (NRF): Non-accelerating & Irrotational Ref. Frame.

***Momentum "P"-Principle*:**

A Single Particle: $\boxed{\underline{F} = m\underline{a} = \dot{\underline{P}}, \quad where: \underline{P} = m\underline{v}}$

A System of Particles: $\boxed{\underline{F} = \sum_{\beta=1}^{N} \dot{\underline{P}}^{\beta} = \dot{\underline{P}}, \quad where: \underline{P} = \sum_{\beta=1}^{N} \underline{P}^{\beta} = \sum_{\beta=1}^{N} m^{\beta} \underline{v}^{\beta} \ (Global - Momentum)}$

A Rigid Body: $\boxed{\underline{F} = m\underline{a}^{C} = \dot{\underline{P}}, \quad where: \underline{P} = m\underline{v}^{C}}$

***Moment-of-Momentum "H"-Principle*:**

A Single Particle: $\boxed{\underline{M}^{A} = \dot{\underline{H}}^{A}, \quad If\ \underline{v}^{A} = 0, or \quad \underline{v}^{A} \| \underline{P}; \quad else: \underline{M}^{A} = \dot{\underline{H}}^{A} + \underline{v}^{A} \times \underline{P}}$ Where: $\boxed{\underline{H}^{A} = \underline{\rho} \times \underline{P}}$

A System of Particles:

$\boxed{\underline{M}^{A} = \dot{\underline{H}}^{A}, \quad If\ \underline{v}^{A} = 0, or \quad A \equiv C, or \quad \underline{v}^{A} \| \underline{P}; \quad else: \underline{M}^{A} = \dot{\underline{H}}^{A} + \underline{v}^{A} \times \underline{P}}$ where: $\boxed{\underline{H}^{A} = \sum_{\beta=1}^{N} \underline{\rho}^{\beta} \times \underline{P}^{\beta}}$

A Rigid Body: **(Euler's Equation):**

$$\boxed{\underline{M}^{A} = \dot{\underline{H}}^{A} = \frac{d}{dt}(\underline{\underline{I}}^{A}.\underline{\omega}), \quad If: \ (A \ \ is \ \ fixed), \ or \ (A \equiv C), \ or \ (\underline{v}^{A} = 0 \ \ or \ \ \underline{v}^{A} \| \underline{P}, and\, \underline{\rho}^{C} \| \underline{a}^{A})}$$

Generalized Forms of the Euler's Equation:

In terms of a rotating coordinate $\{x_i\}$ having an angular velocity $\underline{\Omega}$;

$$\boxed{\underline{M}^{A} = \dot{H}_i^{A} \underline{u}_i + \underline{\Omega} \times \underline{H}^{A} \quad where: H_i^{A} = I_{ij}^{A} \omega_j}$$

If $\{x_i\}$ is fixed to the body (Body Coordinate), then $\underline{\Omega} \to \underline{\omega}$, and $\{I_{ij}^{A}\}$ will form a *constant set*, and;

$$\boxed{\underline{M}^{A} = \underline{\underline{I}}^{A}.\underline{\alpha} + \underline{\omega} \times (\underline{\underline{I}}^{A}.\underline{\omega}) \quad or\, M_i^{A} = I_{ij}^{A} \alpha_j + \gamma_{ijk} \omega_j I_{ks}^{A} \omega_s}$$

In terms of **principal coordinates** at the **mass center**, or **a fixed point A** in a rigid body that is in pure rotation:

$$\boxed{\begin{aligned} M_1^{A} &= I_1^{A} \alpha_1 - (I_2^{A} - I_3^{A}) \omega_2 \omega_3 \\ M_2^{A} &= I_2^{A} \alpha_2 - (I_3^{A} - I_1^{A}) \omega_1 \omega_3 \\ M_3^{A} &= I_3^{A} \alpha_3 - (I_1^{A} - I_2^{A}) \omega_1 \omega_2 \end{aligned}}$$

Inertia Tensor of a System of Particles about a fixed point "O":

$$\boxed{I_{ij}^O = \sum_{\beta=1}^{N} m_\beta [x_k^\beta x_k^\beta \delta_{ij} - x_i^\beta x_j^\beta]}; \text{ or } \boxed{I_{ij}^O = \sum_{\beta=1}^{N} m_\beta \begin{bmatrix} (x_2^\beta)^2 + (x_3^\beta)^2 & -x_1^\beta x_2^\beta & -x_1^\beta x_3^\beta \\ -x_2^\beta x_1^\beta & (x_3^\beta)^2 + (x_1^\beta)^2 & -x_2^\beta x_3^\beta \\ -x_3^\beta x_1^\beta & -x_3^\beta x_2^\beta & (x_1^\beta)^2 + (x_2^\beta)^2 \end{bmatrix}}$$

Rotation Transformation of Inertia Properties:

$$[I_\alpha^O] = [\underline{\underline{T}}][I_{ij}^O][\underline{\underline{T}}^t], \quad \text{where: } \{\underline{e}_\alpha\} \equiv \underline{\underline{T}}\{\underline{e}_i\}, \quad or \quad \{\underline{e}_i\} \equiv \underline{\underline{T}}^t\{\underline{e}_\alpha\}$$

NON-NEWTONIAN REFERENCE FRAME (NNRF): Admissible forces are Newtonian Forces "$\underline{F}$" & Non-Newtonian Coordinate Forces.

***Momentum "$\underline{P}$"-Principle*:**

A Single Particle:

$$\boxed{\overline{\underline{F}} = \frac{d}{dt}(\overline{\underline{P}}) = m\overline{\underline{a}} = \underline{F} - m\underline{a}_o - m\dot{\underline{\Omega}} \times \overline{\underline{r}} - 2m\underline{\Omega} \times \overline{\underline{v}} - m\underline{\Omega} \times (\underline{\Omega} \times \overline{\underline{r}})}$$

A System of Particles & Rigid Body:

$$\boxed{\overline{\underline{F}} = \frac{d}{dt}(\overline{\underline{P}}) = m\overline{\underline{a}}^C = \underline{F} - m\underline{a}_o - m\dot{\underline{\Omega}} \times \overline{\underline{r}}^C - 2m\underline{\Omega} \times \overline{\underline{v}}^C - m\underline{\Omega} \times (\underline{\Omega} \times \overline{\underline{r}}^C)}$$

***Moment-of-Momentum "$\underline{H}$"-Principle*:**

A Single Particle:

$$\boxed{\overline{\underline{M}}^o = \dot{\overline{H}}_i^o \underline{u}_i, or \quad \overline{M}_i^o = \dot{\overline{H}}_i^o}$$

Where: "O" is a fixed point in NNRF, and $\{\underline{u}_i\}$ is fixed to the NNRF.

A System of Particles & Rigid Body:

$$\boxed{\overline{\underline{M}}^o = \dot{\overline{H}}_i^o \underline{u}_i, or \quad \overline{M}_i^o = \dot{\overline{H}}_i^o}$$

Where: $\overline{\underline{M}}^o$ ={Global Moment of Forces}, and

$$\overline{\underline{H}}^o = \sum_{\beta=1}^{N} \overline{\underline{r}}^\beta \times m_\beta \overline{\underline{v}}^\beta = \text{\{Global Moment of Momentum\}}$$

Individual Force Set	Global Forces & Couples	Force Action
$\{-m_\beta \underline{a}_o\}$	$\{-m\underline{a}_o\}$	**They**
$\{-m_\beta \dot{\underline{\Omega}} \times \overline{\underline{r}}^\beta\}$	$\{-m\dot{\underline{\Omega}} \times \overline{\underline{r}}^C, and \quad \overline{\underline{C}} = -\overline{\underline{\underline{I}}}^C \cdot \dot{\underline{\Omega}}\}$	**all act**
$\{-m_\beta \underline{\Omega} \times (\underline{\Omega} \times \overline{\underline{r}}^\beta)\}$	$\{-m\underline{\Omega} \times (\underline{\Omega} \times \overline{\underline{r}}^C), \quad \overline{\underline{C}}^* = -\underline{\Omega} \times (\overline{\underline{\underline{I}}}^C \cdot \underline{\Omega})\}$	**at the**
$\{-2m_\beta \underline{\Omega} \times \overline{\underline{v}}^\beta\}$	$\{-2m\underline{\Omega} \times \overline{\underline{v}}^C, \quad \overline{\underline{C}}^{**} = -\dot{\overline{\underline{\underline{I}}}}^C \cdot \underline{\Omega} - \underline{\Omega} \times \overline{\underline{H}}^C\}$	**mass center.**

ENERGY PRINCIPLES:

***Kinetic Energy*;**

A Single Particle:

$$\boxed{T = \frac{1}{2} m\underline{v}.\underline{v} = \frac{1}{2} m\dot{x}_i \dot{x}_i},$$

A System of Particles:

$$\boxed{T = \frac{1}{2}\sum_{\beta=1}^{N} m_\beta \underline{v}^\beta . \underline{v}^\beta},$$

A Continuum:

$$\boxed{T = \frac{1}{2}\int_m \underline{v}.\underline{v} dm}$$

Total K.E. for a Constant Mass System:

$$\boxed{T = T^{(e)} + T^{(i)} = \frac{1}{2} m\underline{v}^C . \underline{v}^C + \frac{1}{2}\sum_{\beta=1}^{N} m_\beta \dot{\underline{\rho}}^\beta . \dot{\underline{\rho}}^\beta}$$

Total K.E. for a Continuum (Rigid Body):

$$\boxed{T = T^{(e)} + T^{(i)} = \frac{1}{2} m\underline{v}^C . \underline{v}^C + \frac{1}{2}\underline{\omega}.\underline{\underline{I}}^C.\underline{\omega}. = \frac{1}{2}\underline{v}^C . \underline{P} + \frac{1}{2}\underline{\omega}.\underline{H}^C}$$

In terms of Principal Coordinates at "C":

$$\boxed{T^{(i)} = \frac{1}{2}(I_1^C \omega_1^2 + I_2^C \omega_2^2 + I_3^C \omega_3^2)}$$

Work/Power: A Single Particle: $\boxed{U=\int_\Gamma \underline{f}.d\underline{r}}$, Differential Expression: $\boxed{dU=\underline{f}.d\underline{r}}$, Power: $\boxed{\dot{U}=\underline{f}.\underline{v}}$

A System of Particles: $$\boxed{U=U^{(i)}+U^{(e)} \quad or \quad \dot{U}=\dot{U}^{(i)}+\dot{U}^{(e)}=\sum_{\beta=1}^{N}\underline{v}^{\beta}\cdot\underline{f}_{\beta}^{(i)}+\sum_{\beta=1}^{N}\underline{v}^{\beta}\cdot\underline{f}_{\beta}^{(e)}}$$

A Rigid Body with a Body Point "A": $$\boxed{U=U^{(e)} \quad or \quad \dot{U}=\dot{U}^{(e)}=\underline{v}^{A}\cdot\underline{f}_{R}^{(e)}+\underline{\omega}.\underline{M}^{A}}$$

Virtual Work:

For Individual Path of Particles: $\boxed{\delta U=\sum_{\beta=1}^{N}\underline{f}_{\beta}.\delta\underline{r}^{\beta}}$, For a Rigid Body: $\boxed{\delta U=\underline{f}_{R}.\delta\underline{r}^{A}+\underline{M}^{A}.\delta\underline{\theta}}$

LEIBNIZ PRINCIPLE: $\boxed{U=\Delta T, \quad dU=dT, \quad \dot{U}=\dot{T}, \quad \delta U=\delta T}$

LAGRANGIAN EQUATIONS OF MOTION:

For Independent Set of Generalized Coordinates $\{q^m\}$: $\boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^m}\right)-\frac{\partial T}{\partial q^m}=Q_m; \quad m=1,2,...,N}$, or:

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^m}\right)-\frac{\partial L}{\partial q^m}=Q_{NC,m}; \quad m=1,2,...,N; where: \quad L=T-V}$$

For Dependent Set of Generalized Coordinates $\{q^m\}$: $\boxed{g_r(q^m)=0 \quad r=1,...R; \quad m=1,...N}$ = Constraint Functions,

$\boxed{\{C_{rm}\}=\frac{\partial g_r}{\partial q^m}}$ = Constraint Coefficients, $\boxed{C_r^O=\frac{\partial g_r}{\partial t}}$, $\{\lambda^r\}$ = R-number of Lagrangian Multipliers = # of Constraints,

$$\boxed{\begin{array}{l} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^m}\right)-\frac{\partial T}{\partial q^m}+\lambda^r C_{rm}=Q_m; \quad Gives: \quad N-Equations \\ g_r(q^m)=0 \quad or \quad C_{rm}\dot{q}^m+C_r^O=0 \quad Gives: \quad R-Equations \\ \{q^m,\lambda^r\}\Rightarrow N+R \quad Unknowns \end{array}}$$

GENERAL FORM OF HAMILTON's PRINCIPLE:

$$\boxed{\begin{array}{l} \int_{t_0}^{t_1}(\delta U+\delta T)dt=0 \\ \delta\underline{r}(t_0)=\delta\underline{r}(t_1)=\underline{0} \end{array}}$$