Convection Heat Transfer

Boundary Layer Flow
Convection Heat Transfer

Summary of previous session

✓ Laminar Boundary Layer Flow: Flow (Velocity and Temperature fields)

✓ This session: Laminar Boundary Layer Flow: Integral Solution
The next step in the sequence of refining the answers to the friction and heat transfer questions amounts to determining the numerical coefficients (factors) missing from the scaling laws.

In the realm of scale analysis, we made no distinction between the local values of $r$ and $h$ (the values right at $x = L$) and the average values $\tau_{0-L}$ and $h_{0-L}$ defined as:

$$\tau_{0-L} = \frac{1}{L} \int_0^L \tau \, dx, \quad h_{0-L} = \frac{1}{L} \int_0^L h \, dx$$

(46)
In the integral method, we recognize that what we need is not a complete solution for the velocity \( u(x,y) \) and temperature \( T(x,y) \) near the wall, but only the gradients \( \partial (u,T) / \partial y \) evaluated at \( y = 0 \).

We have the opportunity to simplify the boundary layer equations by eliminating \( y \) as a variable.

This is accomplished by integrating each equation term by term from \( y = 0 \) to \( y = Y \), where \( Y > \max (\delta, \delta_T) \) is situated in the free stream.
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\[ u \times \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \right) \]

\[ T \times \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \right) \]

\[ u \left( \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{1}{\rho} \frac{dP}{dx} + \nu \frac{\partial^2 u}{\partial y^2} \quad (26) \]

\[ u \left( \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = \alpha \frac{\partial^2 T}{\partial y^2} \quad (27) \]

\[ \frac{\partial}{\partial x} (u^2) + \frac{\partial}{\partial y} (uv) = -\frac{1}{\rho} \frac{dP}{dx} + \nu \frac{\partial^2 u}{\partial y^2} \quad (47) \]

\[ \frac{\partial}{\partial x} (uT) + \frac{\partial}{\partial y} (vT) = \alpha \frac{\partial^2 T}{\partial y^2} \quad (48) \]
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\[
\frac{\partial}{\partial x} (u^2) + \frac{\partial}{\partial y} (uv) = - \frac{1}{\rho} \frac{dP}{dx} + \nu \frac{\partial^2 u}{\partial y^2}
\]

(47)

\[
\frac{\partial}{\partial x} (uT) + \frac{\partial}{\partial y} (vT) = \alpha \frac{\partial^2 T}{\partial y^2}
\]

(48)

Integrating Eqs. (47) and (48) from \( y = 0 \) to \( y = Y \), and using Leibnitz’s integral formula, yields:

\[
\frac{d}{dx} \int_0^Y u^2 \, dy + u_Y v_y - u_0 v_0 = - \frac{1}{\rho} Y \frac{dP}{dx} + \nu \left( \frac{\partial u}{\partial y} \right)_Y - \nu \left( \frac{\partial u}{\partial y} \right)_0
\]

(49)

\[
\frac{d}{dx} \int_0^Y uT \, dy + v_Y T_Y - v_0 T_0 = \alpha \left( \frac{\partial T}{\partial y} \right)_Y - \alpha \left( \frac{\partial T}{\partial y} \right)_0
\]

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Since the free stream is uniform, we note that \( \frac{\partial}{\partial y} u = 0 \), \( U_y = U_\infty \), and \( T_y = T_\infty \). Also, since the wall is impermeable, \( v_0 = 0 \), and \( v_y \) by performing the same integral on the continuity Equation (7):

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\]

\[
\frac{d}{dx} \int_0^Y u \ dy + v_y - v_0 = 0
\]

(51)

Substituting \( v_y \) into Eqs. (49) and (50), assuming that \( T_\infty \) is, in general, a function of \( x \) and rearranging the resulting expression, we obtain:

\[
\frac{d}{dx} \int_0^Y u(U_\infty - u) \ dy = \frac{1}{\rho} \frac{dP}{dx} + \frac{dU_\infty}{dx} \int_0^Y u \ dy + \nu \left( \frac{\partial u}{\partial y} \right)_0
\]

(52)

\[
\frac{d}{dx} \int_0^Y u(T_\infty - T) \ dy = \frac{dT}{dx} \int_0^Y u \ dy + \alpha \left( \frac{\partial T}{\partial y} \right)_0
\]

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\[
\frac{d}{dx} \int_0^y u(U_\infty - u) \, dy = \frac{1}{\rho} Y \frac{dP_\infty}{dx} + \frac{dU_\infty}{dx} \int_0^y u \, dy + \nu \left( \frac{\partial u}{\partial y} \right)_0
\]

\[
\frac{d}{dx} \int_0^y u(T_\infty - T) \, dy = \frac{dT_\infty}{dx} \int_0^y u \, dy + \alpha \left( \frac{\partial T}{\partial y} \right)_0
\]

Integral boundary layer equations for momentum and energy

They account for the conservation of momentum and energy not at every point \((x,y)\) as Eqs. (26) and (27), but in every slice of thickness \(dx\) and height \(Y\)
Equations (52) and (53) can also be derived by invoking the x momentum theorem and the first law of thermodynamics.

For example, the momentum Equation (52) represents the following force balance:

Forces acting from left to right on the control volume

\[ M_x = \int_0^Y \rho u^2 \, dy \]  Impulse due to the flow of a stream into the control volume

\[ M_y = U_\infty \, d\bar{m} \]  Impulse due to the flow of fast fluid \((U_\infty)\) into the control volume, at a rate \(d\bar{m}\), where \(\bar{m} = \int_0^Y \rho u \, dy\) is the mass flow rate through the slice of height \(Y\)

\[ P_\infty Y \]  Pressure force
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Forces acting from right to left on the control volume

\[ M_{x+dx} = M_x + (dM_x/dx) \, dx \]

- Reaction force due to flow of a stream out of the control volume

\[ \tau \, dx \]

- Tangential force due to friction

\[ Y[P_\infty + (dP_\infty/dx) \, dx] \]

- Pressure force

Setting the resultant of all these forces equal to zero, we derive Eq. (52).

\[ \frac{d}{dx} \int_0^y u(U_\infty - u) \, dy = \frac{1}{\rho} Y \frac{dP_\infty}{dx} + \frac{dU_\infty}{dx} \int_0^y u \, dy + \nu \left( \frac{\partial u}{\partial y} \right)_0 \]

(52)
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Consider the simplest laminar boundary layer problem—the uniform flow \((U_\infty, P_\infty=\text{constants})\).

To solve for the wall shear stress, let us assume that the shape of the longitudinal velocity profile is described by:

\[
 u = \begin{cases} 
 U_\infty m(n), & 0 \leq n \leq 1 \\
 U_\infty, & 1 \leq n 
\end{cases}
\]

(54)
where m is an unspecified shape function that varies from 0 to 1 and n = y/δ

\[ \frac{d}{dx} \int_0^y u(U_\infty - u) \, dy = \frac{1}{\rho} Y \frac{dP}{dx} + \frac{dU_\infty}{dx} \int_0^y u \, dy + \nu \left( \frac{\partial u}{\partial y} \right)_0 \]

(52)

Substituting this assumption into Eq. (52) and noting that
dP/\,dx = 0 and dU/\,dx = 0 yields a first-order ordinary differential equation for the velocity boundary layer thickness δ(x)

Selection of (a) velocity profile and (b) temperature profile for integral boundary layer analysis.
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- **After substitution, we have:**

  \[
  \frac{d}{dx} \left[ \int_0^1 m(1 - m) \, dm \right] = \frac{\nu}{U_\infty} \left( \frac{dm}{dn} \right)_{n=0}
  \]

  \[\text{(55)}\]

- **The resulting expressions for local boundary layer thickness and skin friction coefficient are:**

  \[
  \frac{\delta}{x} = a_1 \, \text{Re}_x^{-1/2}
  \]

  \[\text{(56)}\]

  \[
  C_{f,x} = \frac{\tau}{\frac{1}{2} \rho U_\infty^2} = a_2 \, \text{Re}_x^{-1/2}
  \]

  \[\text{(57)}\]

- **Where:**

  \[
  a_1 = \left[ \frac{2 (dn/dm)_{n=0}}{\int_0^1 m(1 - m) \, dm} \right]^{1/2}
  \]

  \[\text{(56)'}\]

  \[
  a_2 = \left[ 2 \left( \frac{dn}{dm} \right)_{n=0} \int_0^1 m(1 - m) \, dm \right]^{1/2}
  \]

  \[\text{(57)'}\]
The numerical coefficients $a_1$ and $a_2$ depend on the assumption made for the profile shape function $m$: Table shows that as long as this shape is reasonable, the choice of $m(n)$ does not influence the skin friction result appreciably.

<table>
<thead>
<tr>
<th>Profile Shape</th>
<th>$\frac{\delta}{x} \text{ Re}^{1/2}$</th>
<th>$C_{f,x} \text{ Re}_{x}^{1/2}$</th>
<th>Nu Re$^{-1/2}$ Pr$^{-1/3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m = n$</td>
<td>3.46</td>
<td>0.577</td>
<td>Uniform Temperature</td>
</tr>
<tr>
<td>$m = (n/2) (3 - n^2)$</td>
<td>4.64</td>
<td>0.646</td>
<td>(Pr &gt; 1)</td>
</tr>
<tr>
<td>$m = \sin (m/2)$</td>
<td>4.8</td>
<td>0.654</td>
<td>0.289</td>
</tr>
<tr>
<td>Similarity solution</td>
<td>4.92$^a$</td>
<td>0.664</td>
<td>0.331</td>
</tr>
</tbody>
</table>

$^a$Thickness defined as the $y$ value corresponding to $u/U_x = 0.99$. 
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Heat transfer coefficient information is extracted in a similar method from Eq. (53) with dT/dx = 0. Thus, we assume the temperature profile shapes:

\[
T_0 - T = (T_0 - T_\infty)m(p), \quad 0 \leq p \leq 1
\]
\[
T = T_\infty, \quad 1 \leq p
\]

(58)

where \( p = y/\delta_T \). We assume that:

\[
\frac{\delta_T}{\delta} = \Delta
\]

(59)

where \( \Delta \) is a function of Prandtl number only and \( \delta \) is given by Eq. (56): \( \frac{\delta}{x} = a_1 \text{Re}_x^{1/2} \)

Based on these assumptions and \( \delta_T < \delta \) (high-Pr fluids), the integral energy Equation (53) reduces to:

\[
\frac{d}{dx} \int_0^y u(T_\infty - T) \, dy = \frac{dT_\infty}{dx} \int_0^y u \, dy + \alpha \left( \frac{\partial T}{\partial y} \right)_0
\]

Pr = \frac{2(dml/dp)_{p=0}}{(a_1 \Delta)^2} \left[ \int_0^1 m(p \Delta)[1 - m(p)] \, dp \right]^{-1}

(60)
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Assuming the simplest temperature profile, \( m = p \), Eq.\( (60) \) becomes:

\[
\Delta = \Pr^{-1/3}
\]

(61)

The results usually listed in the literature correspond to the cubic profile:

\[
m = \frac{p}{2} (3 - p^2)
\]

The local heat transfer results listed above are anticipated correctly by the scale Analysis Eqs.\( (44) \) and \( (45) \):

\[
h \sim \frac{k}{L} \Pr^{1/3} \text{Re}^{1/2} \quad \text{(Pr >> 1)}
\]

\[
\text{Nu} \sim \Pr^{1/3} \text{Re}^{1/2} \quad \text{(Pr >> 1)}
\]
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In the case of liquid metals ($\Delta >> 1$), instead of Eq. (60), we obtain:

$$Pr = \frac{2(dm/d\rho)p_e}{(a_1 \Delta)^2} \left[ \int_0^{1/\Delta} m(p \Delta) [1 - m(p)] \, dp + \int_{1/\Delta}^1 [1 - m(p)] \, dp \right]^{-1}$$

(65)

The sum of two integrals stems from the fact that when $\delta_T >> \delta$, immediately next to the wall ($0 < y < \delta$), the velocity is described by the assumed shape cm, whereas for $\delta < y < \delta_T$, the velocity is uniform, $u = \delta_T$ [Eq. (54)]. Since $\Delta$ is much greater than unity, the second integral dominates in Eq. (65). Taking again the simplest profile $m = p$, we obtain:

$$\Delta = \frac{\delta_T}{\delta} = (3Pr)^{-1/2} \quad (Pr \ll 1)$$

(66)

$$\frac{\delta_T}{x} = 2Pr^{-1/2} Re_x^{-1/2} \quad (Pr \ll 1)$$

(67)
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we derive the local heat transfer coefficient:

\[ h = \frac{k}{\delta_T} = \frac{1}{2} \frac{k}{\delta_T} \frac{Pr^{1/2}}{Re^{1/2}} \quad (Pr \ll 1) \]  \hspace{1cm} (68)

or the local Nusselt number

\[ Nu = \frac{h x}{k} = \frac{1}{2} \frac{Pr^{1/2}}{Re^{1/2}} \quad (Pr \ll 1) \]  \hspace{1cm} (69)

These results compare favorably with the scaling laws [Eqs. (37)-(40)]. They also compare favorably with more exact (and expensive) solutions.

\[ \frac{\delta_T}{L} \sim Pe_L^{-1/2} \sim Pr^{-1/2} \frac{Re_L^{-1/2}}{Pe_L^{-1/2}} \]

\[ \frac{\delta_T}{\delta} \sim Pr^{-1/2} \gg 1 \]

\[ h \sim \frac{k}{L} \frac{Pr^{1/2}}{Re^{1/2}} \quad (Pr \ll 1) \]

\[ Nu \sim Pr^{1/2} \frac{Re_L^{1/2}}{Pr^{1/2}} \]
Next session:

\[ \text{Laminar Boundary Layer Flow: Similarity Solution} \]