دانشکده‌مهنـــدسی‌مکانیک

# فصل هفتم: مقدمه ای بر دینامیک سه بعدی اجسام صلب

# دینامیک

ارائه: حسین نجات

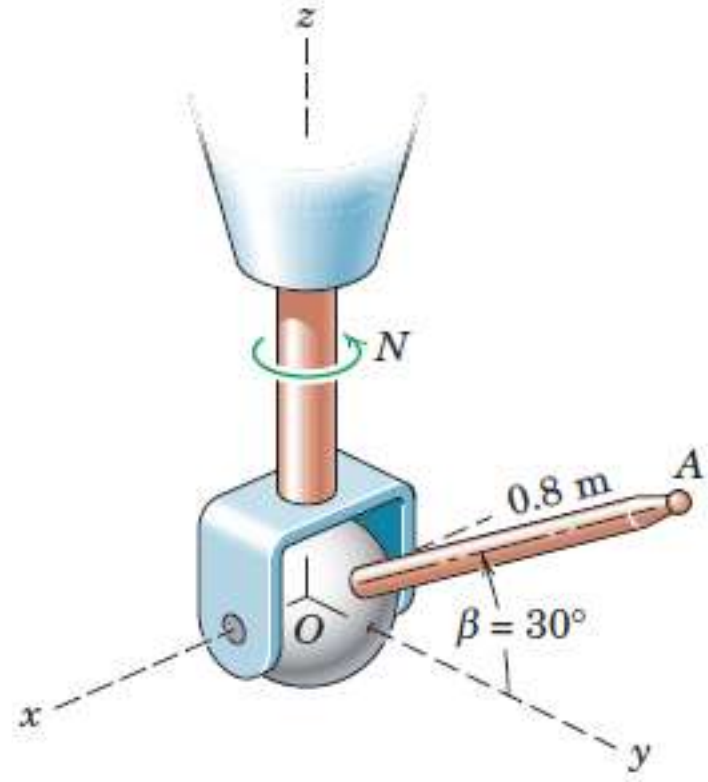
نیمسال اول ۹۴-۱۳۹۳

# رئوس مطالب


۷-۱ مقدمه ای بر سینماتیک سه بعدی

۷-۲ مقدمه ای بر سینتیک سه بعدی

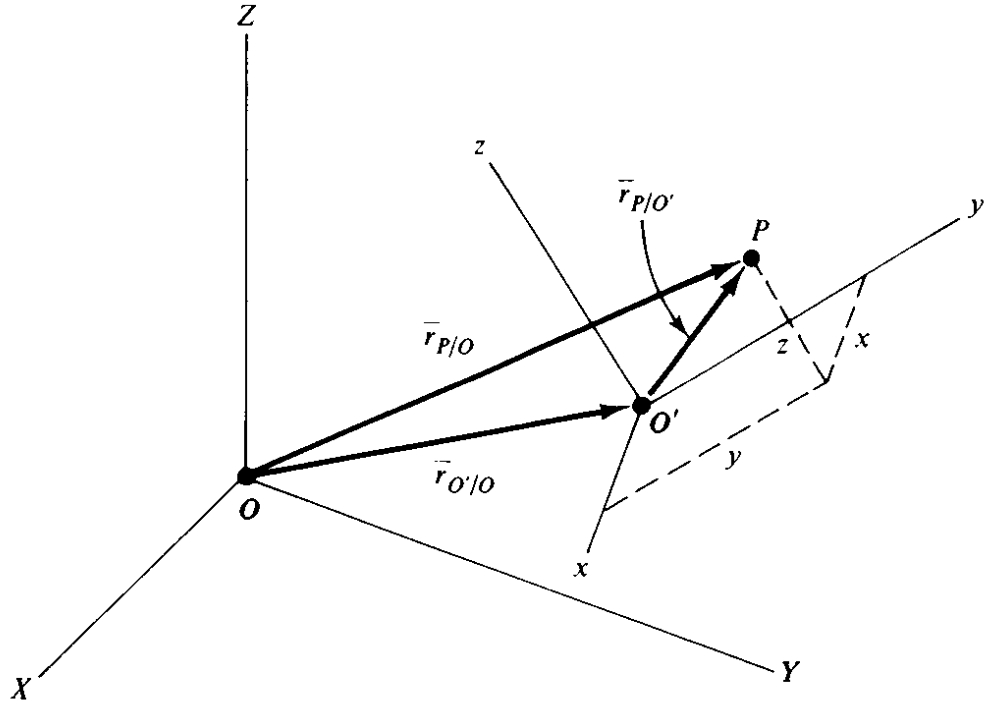
**Example:** The 0.8-m arm *OA* for a remote-control mechanism is pivoted about the horizontal *x*-axis of the clevis, and the entire assembly rotates about the *z*-axis with a constant speed $N$ =60 rev/min. Simultaneously, the arm is being raised at the constant rate $\dot{\beta}$ =4 rad/s. For the position where $\beta = 30$, determine the angular velocity and angular acceleration of *OA*. If, in addition to the motion described, the vertical shaft and point *O* had a linear motion, say, in the *z*-direction, would that motion change the angular velocity or angular acceleration of *OA?*



# VELOCITY AND ACCELERATION USING A MOVING REFERENCE FRAME

We have already seen that the observations of the position of point P from the fixed and moving reference frame are related by

$$\vec{r}_{P/O} = \vec{r}_{O'/O} + \vec{r}_{P/O'}.$$




Differentiating above equation yields: $\vec{v}_P = \vec{v}_{O'} + \dfrac{d}{dt}\vec{r}_{P/O'}.$

Presumably the (moving) xyz reference frame has been chosen for its convenience in describing the position of point P. We therefore describe position $\bar{r}_{P/O'}$ in terms of the coordinates of point P with respect to the axes of this reference frame. Thus,

$$\bar{r}_{P/O'} = x\bar{i} + y\bar{j} + z\bar{k}.$$

$$\Longrightarrow \quad \frac{d}{dt}\bar{r}_{P/O'} = \dot{x}\bar{i} + \dot{y}\bar{j} + \dot{z}\bar{k} + \bar{\omega} \times \bar{r}_{P/O'}.$$

Suppose you were an observer moving with the xyz reference frame; you would see only the (x,y, z) coordinates change. Thus, the first three terms on the right side of above Eq. describe the velocity of point P as seen from the moving reference frame.

$$\boxed{\bar{v}_P = \bar{v}_{O'} + (\bar{v}_P)_{xyz} + \bar{\omega} \times \bar{r}_{P/O'},}$$

where the relative velocity is

$$(\bar{v}_P)_{xyz} = \frac{\delta}{\delta t}\bar{r}_{P/O'} = \dot{x}\bar{i} + \dot{y}\bar{j} + \dot{z}\bar{k}.$$

A relation for the acceleration of a point is obtained by differentiating the velocity relation. The differentiation of second velocity term is:

$$\frac{d}{dt}(\bar{v}_P)_{xyz} = (\bar{a}_P)_{xyz} + \bar{\omega} \times (\bar{v}_P)_{xyz},$$

where the relative acceleration is:

$$(\bar{a}_P)_{xyz} = \frac{\delta}{\delta t}(\bar{v}_P)_{xyz} = \ddot{x}\bar{i} + \ddot{y}\bar{j} + \ddot{z}\bar{k}.$$

The differentiation of third velocity term is:

$$\frac{d}{dt}(\bar{\omega} \times \bar{r}_{P/O'}) = \bar{\alpha} \times \bar{r}_{P/O'} + \bar{\omega} \times [(\bar{v}_P)_{xyz} + \bar{\omega} \times \bar{r}_{P/O'}].$$

The resulting acceleration relation is thereby found to be:

$$\bar{a}_P = \bar{a}_{O'} + (\bar{a}_P)_{xyz} + \bar{\alpha} \times \bar{r}_{P/O'} + \bar{\omega} \times (\bar{\omega} \times \bar{r}_{P/O'}) + 2\bar{\omega} \times (\bar{v}_P)_{xyz}.$$

The final term in above Eq., that preceded by the factor 2, is called the Coriolis acceleration.

**For a merely translating reference frame** xyz, we have:

$$\bar{v}_P = \bar{v}_{O'} + (\bar{v}_P)_{xyz},$$

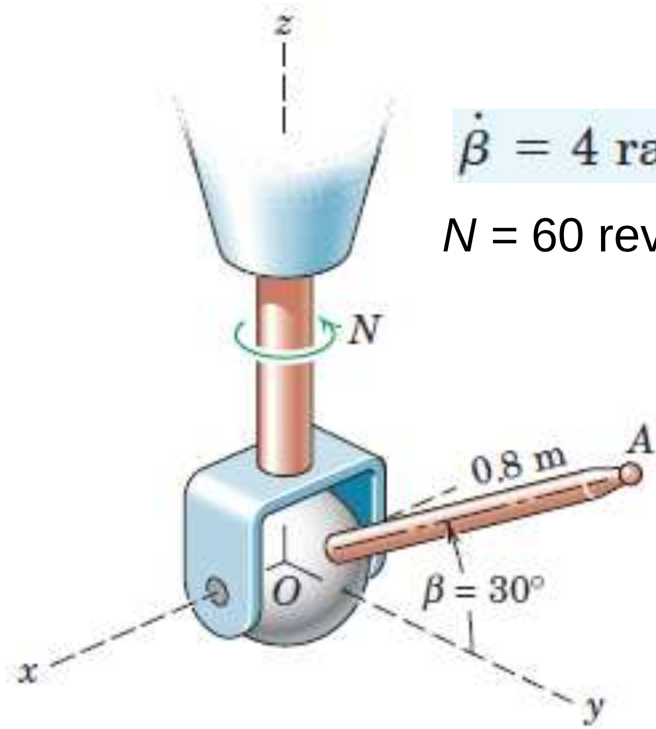
$$\bar{a}_P = \bar{a}_{O'} + (\bar{a}_P)_{xyz}.$$

For a Fixed Particle Relative to xyz

$$\bar{v}_P = \bar{v}_{O'} + \bar{\omega} \times \bar{r}_{P/O'},$$

$$\bar{a}_P = \bar{a}_{O'} + \bar{\alpha} \times \bar{r}_{P/O'} + \bar{\omega} \times (\bar{\omega} \times \bar{r}_{P/O'}).$$

**Example:** Determine velocity and acceleration of point A.




$\dot{\beta} = 4$ rad/s

$N = 60$ rev/min

# سینتیک جسم صلب

- انرژی جنبشی یک جسم صلب به صورت زیر بدست می آید:

$$T = \frac{1}{2} m \bar{v}_G \cdot \bar{v}_G + \frac{1}{2} \bar{\omega} \cdot \bar{H}_G \quad \text{for all motions;}$$

$$T = \frac{1}{2} \bar{\omega} \cdot \bar{H}_O \quad \text{for pure rotation about point } O.$$

- جایی که **H** مومنتوم زاویه ای جسم صلب است و از طریق ماتریس اینرسی با سرعت زاویه ای جسـم صـلب به صورت زیر رابطه دارد:

$$\{H_A\} = [I]\{\omega\},$$

$$[I] = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix}.$$

*inertia matrix*

$$I_{xx} = \iiint (y^2 + z^2)\, dm, \quad I_{yy} = \iiint (x^2 + z^2)\, dm, \quad I_{zz} = \iiint (x^2 + y^2)\, dm$$

$$I_{xy} = \iiint xy\, dm, \quad I_{yz} = \iiint yz\, dm, \quad I_{xz} = \iiint xz\, dm.$$

# معادلات نیوتن-اویلر برای حرکت یک جسم صلب در فضا

- معادلات کلی حرکت جسم صلب در فضا عبارتند از:

$$\sum \bar{F} = m\bar{a}_G,$$

$$\sum \bar{M}_A = \frac{\delta \bar{H}_A}{\delta t} + \bar{\omega} \times \bar{H}_A,$$

$$\bar{H}_A = (I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z)\bar{i} + (I_{yy}\omega_y - I_{xy}\omega_x - I_{yz}\omega_z)\bar{j} + (I_{zz}\omega_z - I_{xz}\omega_x - I_{yz}\omega_y)\bar{k},$$

$$\frac{\delta \bar{H}_A}{\delta t} = (I_{xx}\alpha_x - I_{xy}\alpha_y - I_{xz}\alpha_z)\bar{i} + (I_{yy}\alpha_y - I_{xy}\alpha_x - I_{yz}\alpha_z)\bar{j} + (I_{zz}\alpha_z - I_{xz}\alpha_x - I_{yz}\alpha_y)\bar{k}.$$

- نقطه A مرکز جرم جسم صلب یا یک نقطه ثابت روی جسم است که جسم حول آن دوران خالص دارد.

## فرم ماتریسی معادلات نیوتن-اویلر

$$\begin{Bmatrix} \sum F_x \\ \sum F_y \\ \sum F_z \end{Bmatrix} = m \begin{Bmatrix} a_{Gx} \\ a_{Gy} \\ a_{Gz} \end{Bmatrix},$$

$$\begin{Bmatrix} \sum M_{Ax} \\ \sum M_{Ay} \\ \sum M_{Az} \end{Bmatrix} = [I] \begin{Bmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{Bmatrix} + \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} [I] \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix}.$$

هنگامی که محورهای دستگاه متصل به جسم محورهای اصلی باشند داریم:

$$\sum M_{Ax} = I_{xx}\alpha_x - (I_{yy} - I_{zz})\omega_y\omega_z,$$

$$\sum M_{Ay} = I_{yy}\alpha_y - (I_{zz} - I_{xx})\omega_x\omega_z,$$

$$\sum M_{Az} = I_{zz}\alpha_z - (I_{xx} - I_{yy})\omega_x\omega_y.$$

# مسائل حرکت صفحه ای

- The principles governing spatial motion provide an interesting perspective for the kinetics of planar motion. Let $x$ and $y$ represent convenient directions in the plane, so that $\omega$ is parallel to the $z$ axis. Then

$$\bar{a}_G = a_{Gx}\bar{i} + a_{Gy}\bar{j}, \qquad \bar{\omega} = \omega\bar{k}, \qquad \bar{\alpha} = \dot{\omega}\bar{k}.$$

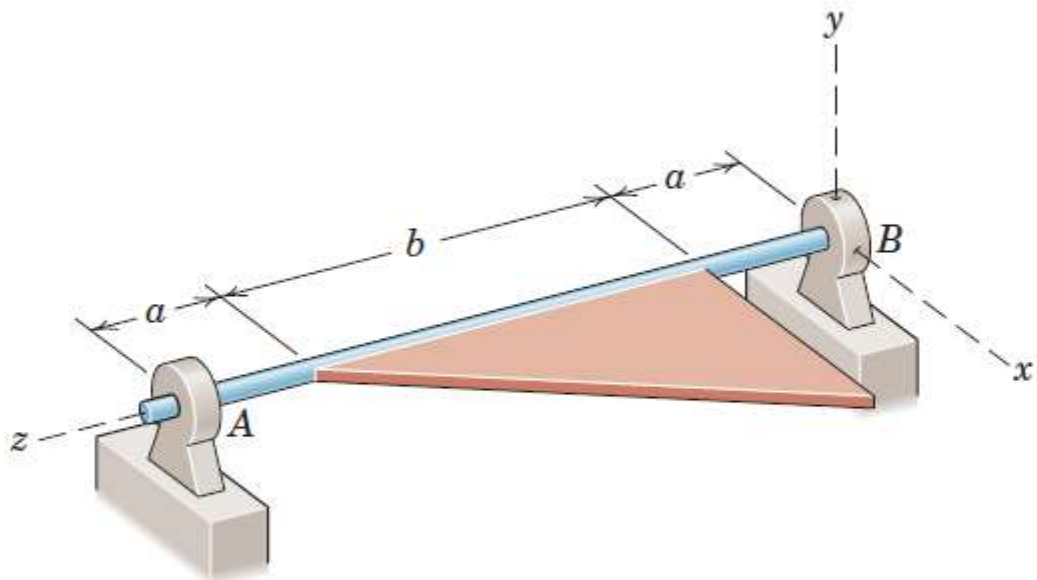
- The corresponding angular momentum for a coordinate system whose origin is at an allowable point is

$$\bar{H}_A = -I_{xz}\omega\bar{i} - I_{yz}\omega\bar{j} + I_{zz}\omega\bar{k}.$$

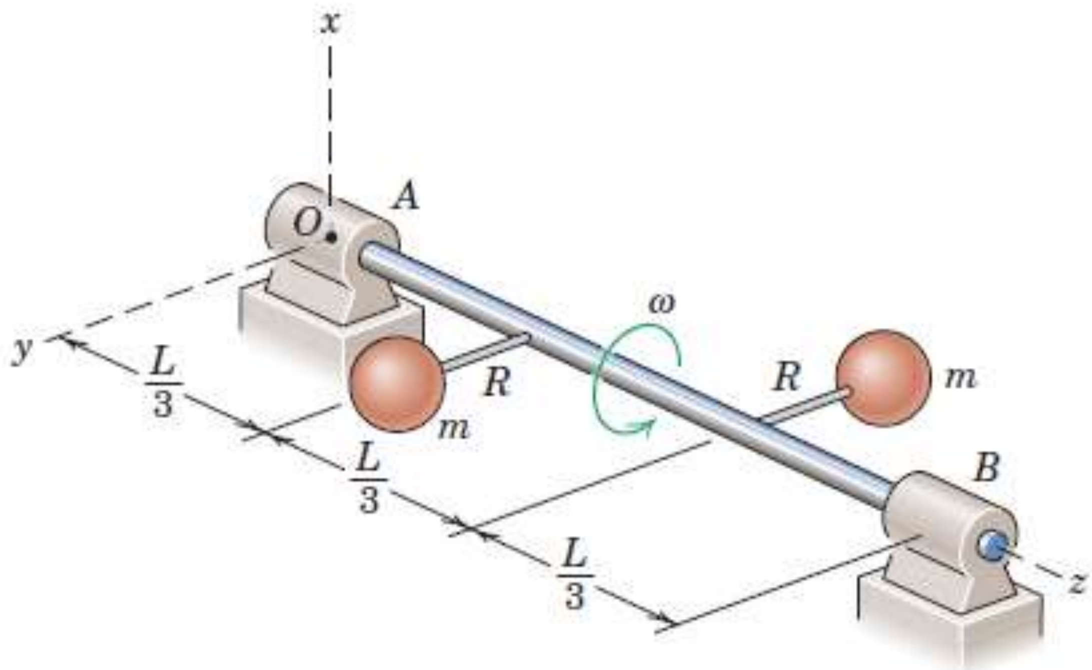
- Hence, the equations of motion are:

$$\sum F_x = ma_{Gx}, \quad \sum F_y = ma_{Gy}, \quad \sum F_z = 0;$$
$$\sum M_{Ax} = -I_{xz}\dot{\omega} + I_{yz}\omega^2, \quad \sum M_{Ay} = -I_{yz}\dot{\omega} + I_{xz}\omega^2, \quad \sum M_{Az} = I_{zz}\dot{\omega}.$$

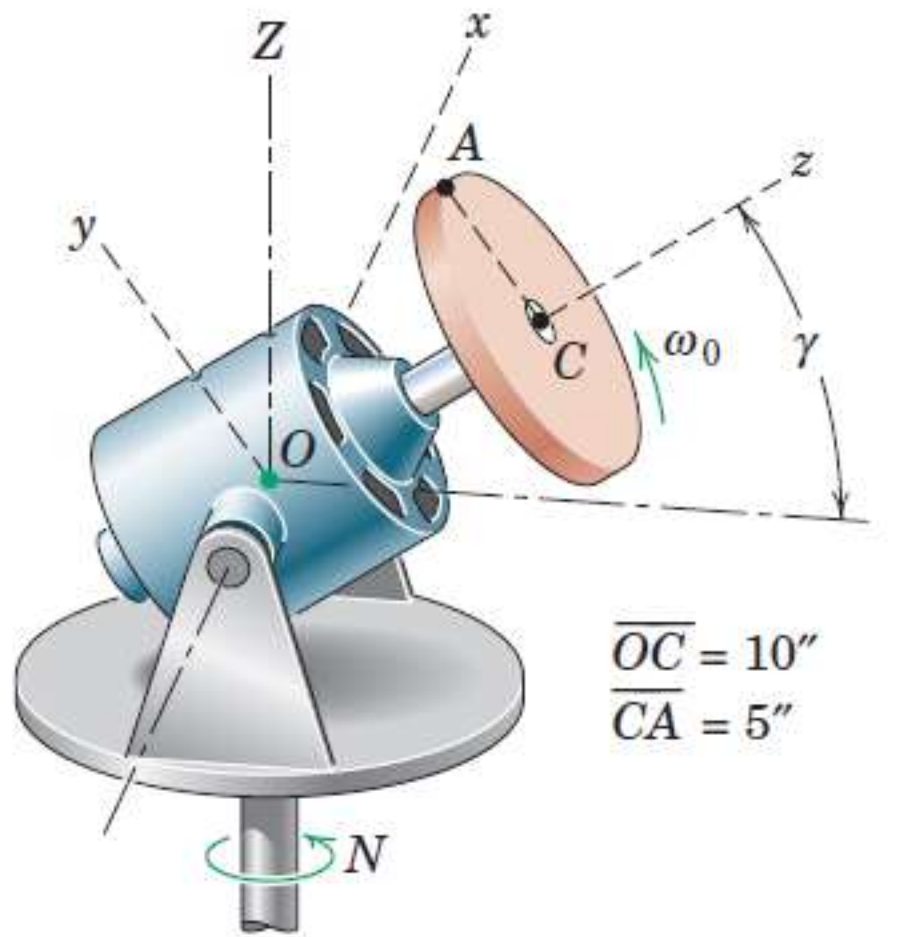
**Example:** The homogeneous thin triangular plate of mass m is welded to the horizontal shaft, which rotates freely in the bearings at A and B. If the plate is released from rest in the horizontal position shown, determine the magnitude of the bearing reaction at A for the instant just after release.



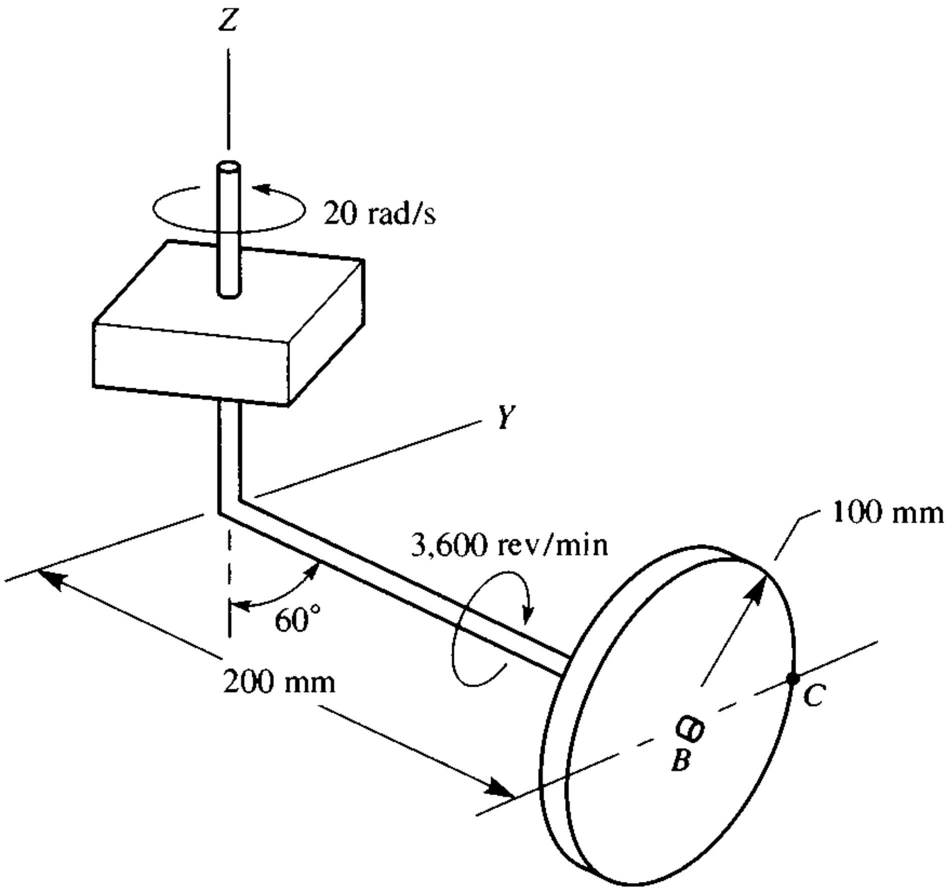
**Example:** The slender shaft carries two offset particles, each of mass $m$, and rotates about the $z$-axis with the constant angular rate $\omega$ as indicated. Determine the $x$- and $y$-components of the bearing reactions at $A$ and $B$ due to the dynamic imbalance of the shaft for the position shown.

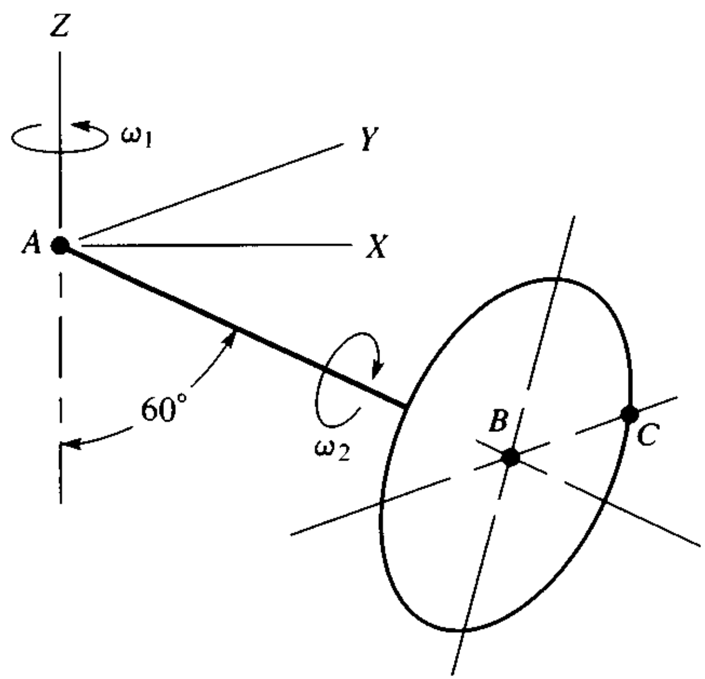


**Example:** The electric motor with an attached disk is running at a constant low speed of 120 rev/min in the direction shown. Its housing and mounting base are initially at rest. The entire assembly is next set in rotation about the vertical $Z$-axis at the constant rate $N$ = 60 rev/min with a fixed angle $\gamma$ of 30. Determine the angular velocity and angular acceleration of disk.



**Example:** The disk is rotating about shaft AB at 3,600 rev/min as the system rotates about the vertical axis at 20 rad/s. Determine the angular velocity of the disk. Use this angular velocity to determine the approximate displacement of point C on the perimeter of the disk 2 microseconds after the instant depicted in the sketch.



We obtain the angular velocity by vectorially adding the rotation rates.

$$\bar{\omega} = \omega_1 \bar{K} + \omega_2 \bar{e}_{A/B},$$

$$\bar{e}_{A/B} = -(\sin 60°)\bar{I} + (\cos 60°)\bar{K}.$$

$$\bar{\omega} = 20\bar{K} + 3{,}600\left(\frac{2\pi}{60}\right)(-0.8660\bar{I} + 0.50\bar{K})$$
$$= -326.5\bar{I} + 208.5\bar{K} \text{ rad/s.}$$

Since Point A is fixed, we can write: $\Delta\bar{r}_C \approx (\bar{\omega} \times \bar{r}_{C/A})\Delta t,$

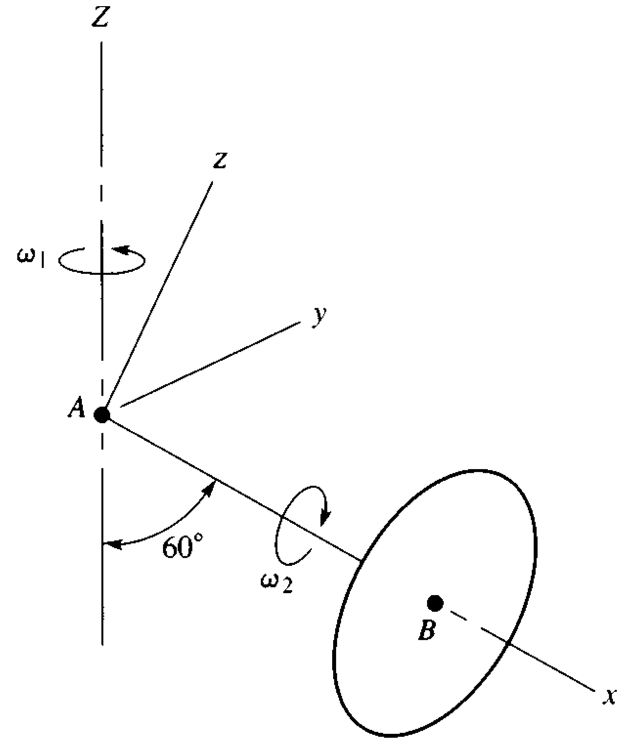
At this instant, the position is

$$\bar{r}_{C/A} = 0.2[(\sin 60°)\bar{I} - (\cos 60°)\bar{K}] + 0.10\bar{J}$$
$$= 0.17321\bar{I} + 0.10\bar{J} - 0.10\bar{K} \text{ m.}$$

Thus, for $\Delta t = 10^{-6} s$, we find

$$\Delta\bar{r}_{C/A} \approx (-326.5\bar{I} + 208.5\bar{K}) \times (0.17321\bar{I} + 0.10\bar{J} - 0.10\bar{K})[2(10^{-6})]$$
$$= (-41.70\bar{I} + 6.929\bar{J} - 65.30\bar{K})(10^{-6}) \text{ m.}$$

**Example:** Determine the angular acceleration of the disk in previous example.




**Solution:** The corresponding general description of the angular velocity is

$$\bar{\omega} = \omega_1 \bar{K} - \omega_2 \bar{i},$$

$$\dot{\bar{K}} = \bar{0} \quad \text{and} \quad \dot{\bar{i}} = \bar{\omega} \times \bar{i}$$

$$\Rightarrow \quad \bar{\alpha} = -\omega_2 (\bar{\omega} \times \bar{i}).$$

$$\bar{K} = -(\cos 60°)\bar{i} + (\sin 60°)\bar{k}.$$

$$\bar{\omega} = 20(-0.50\bar{i} + 0.866\bar{k}) - 120\pi\bar{i} = -387.0\bar{i} + 17.32\bar{k} \text{ rad/s},$$

$$\bar{\alpha} = -(120\pi)(-387.0\bar{i} + 17.32\bar{k}) \times \bar{i} = -6{,}530\bar{j} \text{ rad/s}^2.$$