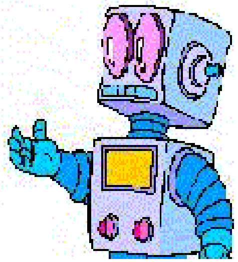
INTRODUCTION TO ROBOTICS
(Kinematics, Dynamics, and Design)

SESSION # 17:
MANIPULATOR DYNAMICS



Ali Meghdari, Professor

School of Mechanical Engineering

Sharif University of Technology

Tehran, IRAN 11365-9567

Homepage: http://meghdari.sharif.edu

Manipulator Dynamics

➢ Iterative Newton-Euler Dynamic Algorithm:

First: Compute link velocities and accelerations iteratively from link-1 to link-n, and apply the Newton-Euler equations to each link.

Second: Compute the forces and torques of interaction recursively from link-n back to link-1.

Outward iterations: $i : 0 \rightarrow 5$

$$ {}^{i+1}\omega_{i+1} = {}^{i+1}_{\ \ i}R\,{}^{i}\omega_i + \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}, \tag{6.45}$$

$$ {}^{i+1}\dot{\omega}_{i+1} = {}^{i+1}_{\ \ i}R\,{}^{i}\dot{\omega}_i + {}^{i+1}_{\ \ i}R\,{}^{i}\omega_i \times \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} + \ddot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}, \tag{6.46}$$

$$ {}^{i+1}\dot{v}_{i+1} = {}^{i+1}_{\ \ i}R\left({}^{i}\dot{\omega}_i \times {}^{i}P_{i+1} + {}^{i}\omega_i \times \left({}^{i}\omega_i \times {}^{i}P_{i+1}\right) + {}^{i}\dot{v}_i\right), \tag{6.47}$$

$$ \begin{aligned} {}^{i+1}\dot{v}_{C_{i+1}} &= {}^{i+1}\dot{\omega}_{i+1} \times {}^{i+1}P_{C_{i+1}} \\ &\quad + {}^{i+1}\omega_{i+1} \times \left({}^{i+1}\omega_{i+1} \times {}^{i+1}P_{C_{i+1}}\right) + {}^{i+1}\dot{v}_{i+1}, \end{aligned} \tag{6.48}$$

$$ {}^{i+1}F_{i+1} = m_{i+1}\,{}^{i+1}\dot{v}_{C_{i+1}}, \tag{6.49}$$

$$ {}^{i+1}N_{i+1} = {}^{C_{i+1}}I_{i+1}\,{}^{i+1}\dot{\omega}_{i+1} + {}^{i+1}\omega_{i+1} \times {}^{C_{i+1}}I_{i+1}\,{}^{i+1}\omega_{i+1}. \tag{6.50}$$

Inward iterations: $i : 6 \rightarrow 1$

$$ {}^{i}f_i = {}^{\ \ i}_{i+1}R\,{}^{i+1}f_{i+1} + {}^{i}F_i, \tag{6.51}$$

$$ \begin{aligned} {}^{i}n_i &= {}^{i}N_i + {}^{\ \ i}_{i+1}R\,{}^{i+1}n_{i+1} + {}^{i}P_{C_i} \times {}^{i}F_i \\ &\quad + {}^{i}P_{i+1} \times {}^{\ \ i}_{i+1}R\,{}^{i+1}f_{i+1}, \end{aligned} \tag{6.52}$$

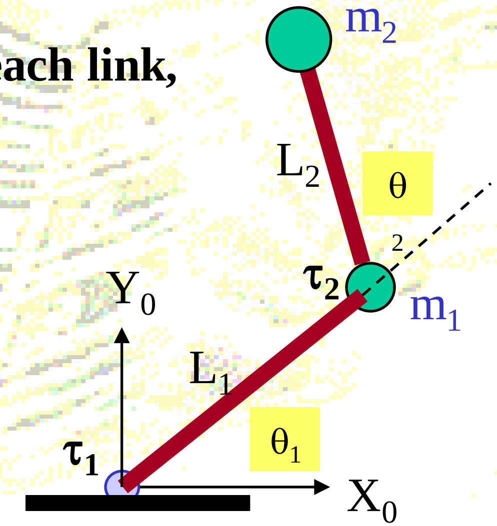
$$ \tau_i = {}^{i}n_i^T\,{}^{i}\hat{Z}_i. \tag{6.53}$$

Manipulator Dynamics

➢ **Closed-form (Symbolic Form) Dynamic Equations:**

Example: The 2-DOF Manipulator Arm.

Assumptions: Point masses at the distal end of each link,



$$^{0}\dot{v}_0 = g\hat{Y}_0 = \begin{bmatrix} 0 \\ g \\ 0 \end{bmatrix}, \qquad (gravity-term)$$

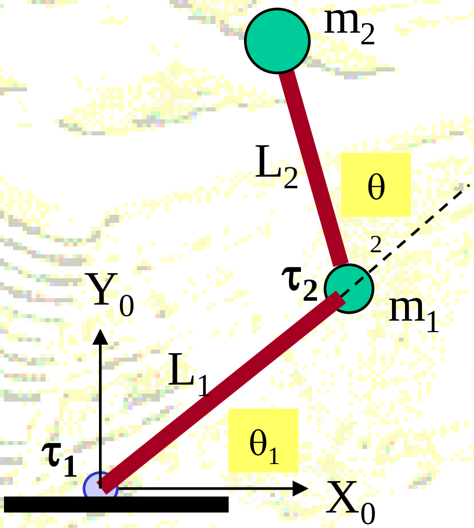
$$\left\{ \begin{matrix} {}^{C1}I_1 = 0 \\ {}^{C2}I_2 = 0 \end{matrix} \right\} (point-mass)$$

$$\tau_1 = m_2\ell_2^2(\ddot{\theta}_1 + \ddot{\theta}_2) + m_2\ell_1\ell_2C_2(2\ddot{\theta}_1 + \ddot{\theta}_2) + (m_1 + m_2)\ell_1^2\ddot{\theta}_1 -$$
$$m_2\ell_1\ell_2S_2\dot{\theta}_2^2 - 2m_2\ell_1\ell_2S_2\dot{\theta}_1\dot{\theta}_2 + m_2\ell_2gC_{12} + (m_1 + m_2)\ell_1gC_1$$
$$\tau_2 = m_2\ell_1\ell_2C_2\ddot{\theta}_1 + m_2\ell_1\ell_2S_2\dot{\theta}_1^2 + m_2\ell_2gC_{12} + m_2\ell_2^2(\ddot{\theta}_1 + \ddot{\theta}_2)$$

Actuator torques as a function of joints position, velocity, and acceleration.

Manipulator Dynamics

- **The Structure of Dynamic Equations**



The State-Space Equation:

$$\tau = M(\theta)\ddot{\theta} + V(\theta,\dot{\theta}) + G(\theta)$$

Where:

M(θ): **Mass Matrix of the Manipulator (always symmetric & non-singular)**

$$M(\theta) = \begin{bmatrix} m_2\ell_2^2 + 2m_2\ell_1\ell_2C_2 + (m_1+m_2)\ell_1^2 & m_2\ell_2^2 + m_2\ell_1\ell_2C_2 \\ m_2\ell_2^2 + m_2\ell_1\ell_2C_2 & m_2\ell_2^2 \end{bmatrix}$$

V(θ, θ_{dot}): **The Velocity Terms**

$$V(\theta,\dot{\theta}) = \begin{bmatrix} -m_2\ell_1\ell_2S_2\dot{\theta}_2^2 - 2m_2\ell_1\ell_2S_2\dot{\theta}_1\dot{\theta}_2 \\ m_2\ell_1\ell_2S_2\dot{\theta}_1^2 \end{bmatrix}$$

Manipulator Dynamics

- **The Structure of Dynamic Equations**

G(θ): The Gravity Term

$$G(\theta)=\begin{bmatrix} m_2\ell_2 gC_{12}+(m_1+m_2)\ell_1 gC_1 \\ m_2\ell_2 gC_{12} \end{bmatrix}$$

Including other effects:

F(θ, θdot): The Friction Terms (may also be a function of position θ as well)

$$Viscous \equiv \tau_f = v\dot{\theta}$$

$$Coulomb \equiv \tau_f = C\,\mathrm{sgn}(\dot{\theta}) = \begin{cases} C = X \text{ when } \dot{\theta}=0 \Leftrightarrow Static \\ C = Y \text{ when } \dot{\theta}\neq 0 \Leftrightarrow Dynamic, Y < X \end{cases}$$

v = viscous, and C = Coulomb friction coefficients

A reasonable model: $\tau_{friction} = v\dot{\theta} + C\,\mathrm{sgn}(\dot{\theta}) \equiv F(\theta,\dot{\theta})$

Manipulator Dynamics

- **The Structure of Dynamic Equations**

Finally;

$$\tau = M(\theta)\ddot{\theta} + V(\theta,\dot{\theta}) + G(\theta) + F(\theta,\dot{\theta})$$

Note that: **we have ignored link flexibility. Only rigid links are considered (Flexibilities are extremely difficult to model).**

Manipulator Dynamics

➢ Lagrangian Formulation of Manipulator Dynamics

- The Newton-Euler's Formulation is a "Force-Balance" Approach to Dynamics.
- The Lagrangian Formulation is an "Energy-Based" approach to Dynamics. We can derive the equations of motion for any *n*-DOF system by using energy methods.
  - All we need to know are the conservative (kinetic and potential) and non-conservative (dissipative) terms

The general form of Lagrangian Equations of motion (*for independent set of generalized coordinates*) for manipulators are:

Manipulator Dynamics

➢ **Lagrangian Formulation of Manipulator Dynamics**

$$F_i = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i}$$

Where:

L (Lagrangian) **= K.E.** (System's Kinetic Energy) **– P.E.** (System's Potential Energy)

q_i: Coordinates in which the Kinetic and Potential energies are expressed. (Generalized Coordinate)

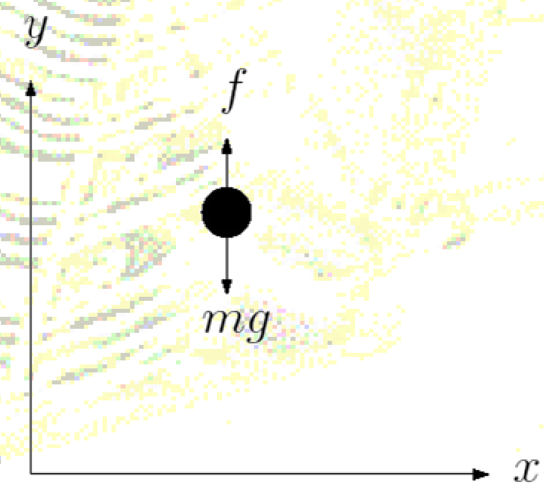
F_i: The corresponding Force or Torque, depending on whether q_i is a linear or angular coordinate. (The Generalized Force)

Manipulator Dynamics

Ex: 1-DOF system

- **Let us derive the equations of motion for a 1-DOF system:**
  - **Consider a particle of mass *m***
  - **Using Newton's second law:**

$$m\ddot{y} = f - mg$$



  - **Now define the kinetic and potential energies:**

$$K = \frac{1}{2}m\dot{y}^2 \qquad P = mgy$$

  - **Rewrite the above differential equation**
    - **Left side:**

$$m\ddot{y} = \frac{d}{dt}(m\dot{y}) = \frac{d}{dt}\frac{\partial}{\partial \dot{y}}\left(\frac{1}{2}m\dot{y}^2\right) = \frac{d}{dt}\frac{\partial K}{\partial \dot{y}}$$

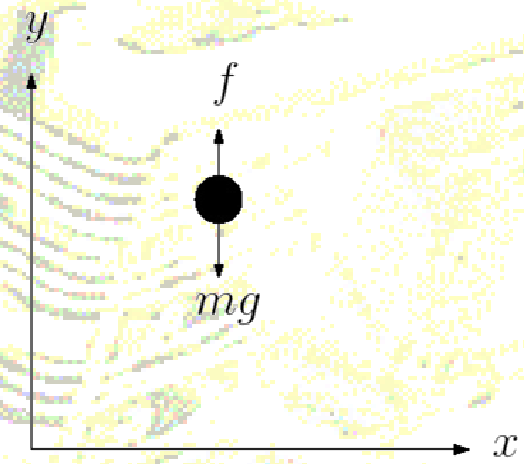
    - **Right side:**

$$mg = \frac{\partial}{\partial y}(mgy) = \frac{\partial P}{\partial y}$$

Manipulator Dynamics

- Thus we can rewrite the initial equation:

$$\frac{d}{dt}\frac{\partial K}{\partial \dot{y}} = f - \frac{\partial P}{\partial y}$$



- Now we make the following definition:

$$L = K - P$$

- *L* is called the "***Lagrangian***"
  - We can rewrite our equation of motion again:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = f$$

- Thus, to define the equation of motion for this system, all we need is a description of the potential and kinetic energies.

Manipulator Dynamics

- If we represent the variables of the system as "generalized coordinates", then we can write the equations of motion for an *n*-DOF system as:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \tau_i$$

- It is important to recognize the form of the above equation:
  – The left side contains the conservative terms
  – The right side contains the non-conservative terms

- This formulation leads to a set of *n* coupled 2nd order differential equations.

$$F_i = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i}$$

Manipulator Dynamics

Ex: 1-DOF system

- **Single link, single motor coupled by a drive shaft:**
  - **θ_m and θ_l are the angular displacements of the shaft and the link respectively, related by a gear ratio, *r*:**

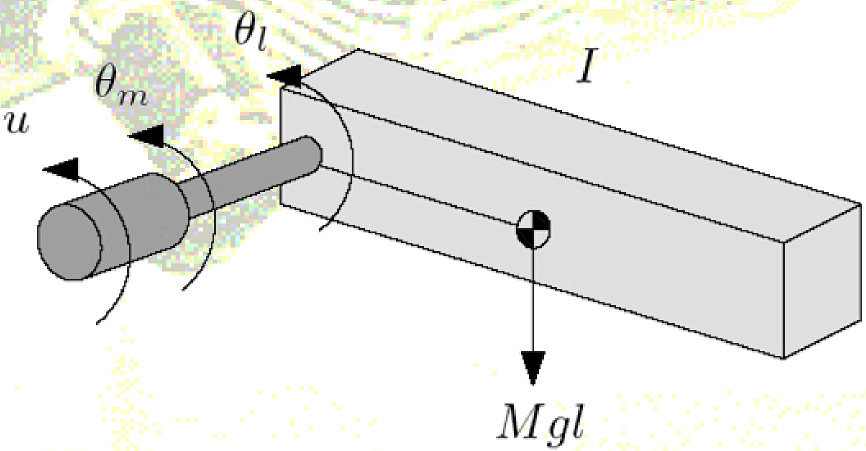
$$\theta_m = r\theta_l$$

  - **Start by determining the kinetic and potential energies:**

$$K = \frac{1}{2}J_m\dot{\theta}_m^2 + \frac{1}{2}J_l\dot{\theta}_l^2$$

$$= \frac{1}{2}\left(r^2 J_m + J_l\right)\dot{\theta}_l^2$$

$$P = \frac{MgL}{2}\left(1 - \cos\theta_l\right)$$



  - **J_m and J_l are the motor/shaft and link inertias respectively and *M* and *L* are the mass and length of the link respectively.**

Manipulator Dynamics

- Let the total inertia, *J*, be defined by: $$J = r^2 J_m + J_l$$
- Now write the Lagrangian: $$L = \frac{1}{2} J \dot{\theta}_l^2 - \frac{MgL}{2}\left(1 - \cos\theta_l\right)$$
- Thus we can write the equation of motion for this 1-DOF system as:

$$J\ddot{\theta}_l + \frac{MgL}{2}\sin\theta_l = \tau_l$$

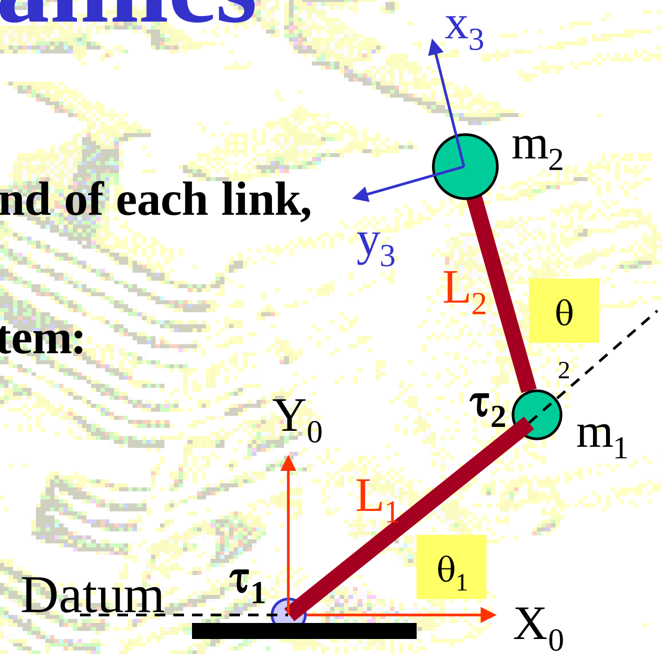
- The right side contains the non-conservative terms such as:
  - The input motor torque: $u = r\tau_m$
  - Damping torques: $B = rB_m + B_l$
- Therefore we can rewrite the equation of motion as:

$$J\ddot{\theta}_l + B\dot{\theta}_l + \frac{MgL}{2}\sin\theta_l = u$$

Manipulator Dynamics

Example: The 2-DOF Manipulator Arm.

Assumptions: Point masses at the distal end of each link,



Compute the Kinetic and Potential Energies of the System:

$$K.E.)_{total} = K.E.)_1 + K.E.)_2$$

$$P.E.)_{total} = P.E.)_1 + P.E.)_2$$

For the mass m_1 we have:

$$K.E.)_1 = \frac{1}{2} m_1 \ell_1^2 \dot{\theta}_1^2$$

$$P.E.)_1 = m_1 g \ell_1 Sin\theta_1$$

For the mass m_2 we have:

$$x_3 = \ell_1 Cos\theta_1 + \ell_2 Cos(\theta_1 + \theta_2)$$

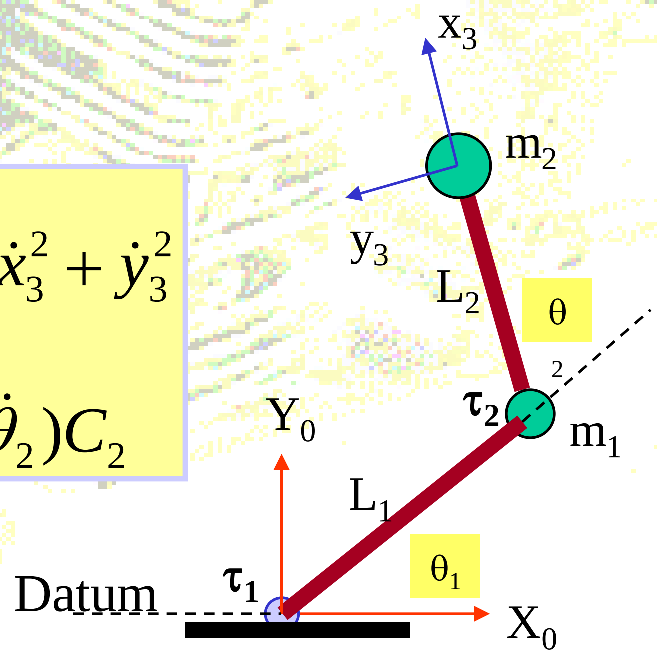
$$y_3 = \ell_1 Sin\theta_1 + \ell_2 Sin(\theta_1 + \theta_2)$$

Manipulator Dynamics

Example: The 2-DOF Manipulator Arm.

Assumptions: Point masses at the distal end of each link,

For the mass m_2 we have:



$$\left\{\begin{aligned}\dot{x}_3 &= -\ell_1\dot{\theta}_1 S_1 - \ell_2 S_{12}(\dot{\theta}_1+\dot{\theta}_2)\\ \dot{y}_3 &= \ell_1\dot{\theta}_1 C_1 + \ell_2 C_{12}(\dot{\theta}_1+\dot{\theta}_2)\end{aligned}\right\} \Rightarrow v_3^2 = \dot{x}_3^2 + \dot{y}_3^2$$

$$v_3^2 = \ell_1^2\dot{\theta}_1^2 + \ell_2^2(\dot{\theta}_1+\dot{\theta}_2)^2 + 2\ell_1\ell_2\dot{\theta}_1(\dot{\theta}_1+\dot{\theta}_2)C_2$$

$$K.E.)_2 = \frac{1}{2} m_2 v_3^2$$

$$P.E.)_2 = m_2 g y_3 = m_2 g \ell_1 S_1 + m_2 g \ell_2 S_{12}$$

Therefore: $\mathbf{L = K.E.)_{sys.} - P.E.)_{sys.}}$

Manipulator Dynamics

Therefore: **$L = K.E.)_{sys.} - P.E.)_{sys.}$**

$$L = [\frac{1}{2}(m_1 + m_2)\ell_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2\ell_2^2(\dot{\theta}_1 + \dot{\theta}_2)^2 + m_2\ell_1\ell_2C_2\dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_2)]$$
$$-[(m_1 + m_2)g\ell_1S_1 + m_2g\ell_2S_{12}]$$

For $q_i = \theta_1$, we have:

$$\frac{\partial L}{\partial \dot{\theta}_1} = (m_1 + m_2)\ell_1^2\dot{\theta}_1 + m_2\ell_2^2(\dot{\theta}_1 + \dot{\theta}_2) + 2m_2\ell_1\ell_2C_2\dot{\theta}_1 + m_2\ell_1\ell_2C_2\dot{\theta}_2$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} = [(m_1 + m_2)\ell_1^2 + m_2\ell_2^2 + 2m_2\ell_1\ell_2C_2]\ddot{\theta}_1 + [m_2\ell_2^2 + m_2\ell_1\ell_2C_2]\ddot{\theta}_2$$
$$-2m_2\ell_1\ell_2S_2\dot{\theta}_1\dot{\theta}_2 - m_2\ell_1\ell_2S_2\dot{\theta}_2^2$$

Manipulator Dynamics

Therefore: **L = K.E.)$_{sys.}$ – P.E.)$_{sys.}$**

$$\frac{\partial L}{\partial \theta_1} = -(m_1 + m_2) g \ell_1 C_1 - m_2 g \ell_2 C_{12}$$

For $q_i = \theta_1$, we have:

$$\tau_1 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} - \frac{\partial L}{\partial \theta_1}$$

$$\tau_1 = m_2 \ell_2^2 (\ddot{\theta}_1 + \ddot{\theta}_2) + m_2 \ell_1 \ell_2 C_2 (2\ddot{\theta}_1 + \ddot{\theta}_2) + (m_1 + m_2) \ell_1^2 \ddot{\theta}_1 - m_2 \ell_1 \ell_2 S_2 \dot{\theta}_2^2 - 2 m_2 \ell_1 \ell_2 S_2 \dot{\theta}_1 \dot{\theta}_2 + m_2 \ell_2 g C_{12} + (m_1 + m_2) \ell_1 g C_1$$

Manipulator Dynamics

$$L = [\frac{1}{2}(m_1 + m_2)\ell_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2\ell_2^2(\dot{\theta}_1 + \dot{\theta}_2)^2 + m_2\ell_1\ell_2 C_2\dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_2)]$$
$$-[(m_1 + m_2)g\ell_1 S_1 + m_2 g\ell_2 S_{12}]$$

For $q_i = \theta_2$, we have:

$$\frac{\partial L}{\partial \dot{\theta}_2} = m_2\ell_2^2(\dot{\theta}_1 + \dot{\theta}_2) + m_2\ell_1\ell_2 C_2\dot{\theta}_1$$

$$\frac{\partial L}{\partial \theta_2} = -m_2\ell_1\ell_2 S_2\dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_2) - m_2 g\ell_2 C_{12}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} = m_2\ell_2^2(\ddot{\theta}_1 + \ddot{\theta}_2) + m_2\ell_1\ell_2 C_2\ddot{\theta}_1 - m_2\ell_1\ell_2 S_2\dot{\theta}_1\dot{\theta}_2$$

$$\tau_2 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} - \frac{\partial L}{\partial \theta_2}$$

$$\tau_2 = m_2\ell_1\ell_2 C_2\ddot{\theta}_1 + m_2\ell_1\ell_2 S_2\dot{\theta}_1^2 + m_2\ell_2 g C_{12} + m_2\ell_2^2(\ddot{\theta}_1 + \ddot{\theta}_2)$$

Manipulator Dynamics

- **Formulating Dynamic Equations in Cartesian Space**

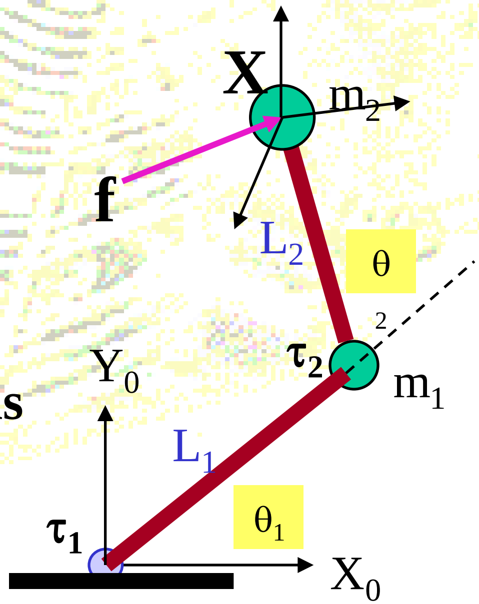
In Joint Space: The General form of Dynamic Equations is:

$$\tau = M(\theta)\ddot{\theta} + V(\theta,\dot{\theta}) + G(\theta)$$

Where:

τ: The Vector of Joint Torques

θ: The Vector of Joint Variables



Sometimes it is important to have the Dynamic Equations in Cartesian Space as:

$$f = M_x(\theta)\ddot{X} + V_x(\theta,\dot{\theta}) + G_x(\theta)$$

Where:

f: The Force-Torque acting at the tip of the arm

X: A Cartesian Vector representing position & orientation of the end-effector

Manipulator Dynamics

- **Formulating Dynamic Equations in Cartesian Space**

In Cartesian Space:

$$f = M_x(\theta)\ddot{X} + V_x(\theta,\dot{\theta}) + G_x(\theta)$$

Where:

f: The Force-Torque acting at the tip of the arm

X: A Cartesian Vector representing position & orientation of the end-effector

$M_x(\theta)$: Cartesian Mass Matrix

$V_x(\theta)$: Vector of Velocity Terms in Cartesian Space

$G_x(\theta)$: Gravity Terms in Cartesian Space

To obtain Dynamic Equations in Cartesian Space, we have:

Manipulator Dynamics

- **Formulating Dynamic Equations in Cartesian Space**

$$\tau = J^T(\theta) f \Rightarrow J^{-T}\tau = f$$

Note that:

$$\tau = M(\theta)\ddot{\theta} + V(\theta,\dot{\theta}) + G(\theta)$$

Pre-multiplying J^{-T} on the above equation:

$$J^{-T}\tau = J^{-T}M(\theta)\ddot{\theta} + J^{-T}V(\theta,\dot{\theta}) + J^{-T}G(\theta) = f$$

But from the definition of Jacobian we have:

$$\dot{X} = J\dot{\theta} \Rightarrow \ddot{X} = \dot{J}\dot{\theta} + J\ddot{\theta} \Rightarrow \ddot{\theta} = J^{-1}\ddot{X} - J^{-1}\dot{J}\dot{\theta}$$

Substituting in Equation (*), we have:

$$f = J^{-T}M(\theta)J^{-1}\ddot{X} - J^{-T}M(\theta)J^{-1}\dot{J}\dot{\theta} + J^{-T}V(\theta,\dot{\theta}) + J^{-T}G(\theta)$$

Manipulator Dynamics

➢ **Formulating Dynamic Equations in Cartesian Space**

$$f = J^{-T}M(\theta)J^{-1}\ddot{X} - J^{-T}M(\theta)J^{-1}\dot{J}\dot{\theta} + J^{-T}V(\theta,\dot{\theta}) + J^{-T}G(\theta)$$

$$M_x(\theta) = J^{-T}M(\theta)J^{-1}$$

$$V_x(\theta,\dot{\theta}) = J^{-T}[V(\theta,\dot{\theta}) - M(\theta)J^{-1}\dot{J}\dot{\theta}]$$

$$G_x(\theta) = J^{-T}G(\theta)$$

Where:

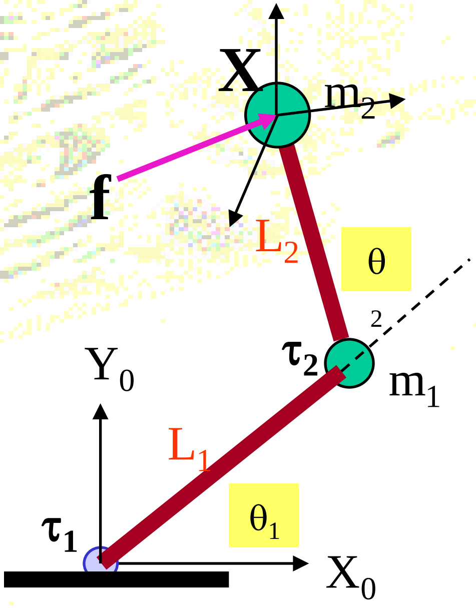
J: Jacobian written in the same frame as f and X.

Manipulator Dynamics

Example: **The 2-DOF Manipulator Arm.**

$$J(\theta)=\begin{bmatrix}\ell_1 S_2 & 0\\ \ell_1 C_2+\ell_2 & \ell_2\end{bmatrix}_1 \Rightarrow J^{-1}=\frac{1}{\ell_1\ell_2 S_2}\begin{bmatrix}\ell_2 & 0\\ -\ell_1 C_2-\ell_2 & \ell_1 S_2\end{bmatrix}$$

$$\dot{J}(\theta)=\begin{bmatrix}\ell_1 C_2\dot{\theta}_2 & 0\\ -\ell_1 S_2\dot{\theta}_2 & 0\end{bmatrix}_1$$



$M_x(\theta)$, $V_x(\theta)$, $G_x(\theta)$ are found as follows:

$$M_x(\theta)=\begin{bmatrix}m_2+\dfrac{m_1}{S_2} & 0\\ 0 & m_2\end{bmatrix}$$

$$V_x(\theta)=\begin{bmatrix}\cdots\\ \cdots\end{bmatrix},\qquad G_x(\theta)=\begin{bmatrix}m_1 g\dfrac{C_1}{S_2}+m_2 g S_{12}\\ m_2 g C_{12}\end{bmatrix}$$

Manipulator Dynamics

- **Dynamic Simulation:** **Given the vector of joint torques, compute the resulting motion of the arm (forward dynamic).**

To simulate the motion of a manipulator arm, we need the dynamic equations as:

$$\tau = M(\theta)\ddot{\theta} + V(\theta,\dot{\theta}) + G(\theta) + F(\theta,\dot{\theta})$$

Solve for;

$$\ddot{\theta} = M^{-1}(\theta)[\tau - V(\theta,\dot{\theta}) - G(\theta) - F(\theta,\dot{\theta})]$$

Then, integrate to get $\{\dot{\theta},\theta\}$ numerically (Runge-Kutta, Euler Method, etc.), given the initial conditions on the motion of the arm (i.e. $\theta(0)=\theta_0, \dot{\theta}(0)=0, etc.$).

Exercises:

6.1, 6.2, 6.4, 6.5

Programming Exercises:

6.1, 6.2

MATLAB Exercise: 6A

Programming Exercises: 6.1, 6.2

Robotic Project.exe