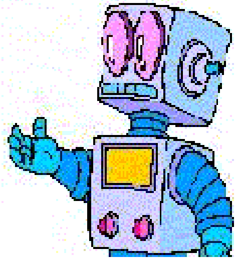
INTRODUCTION TO ROBOTICS

(Kinematics, Dynamics, and Design)

SESSION # 15:

MANIPULATOR'S JACOBIANS



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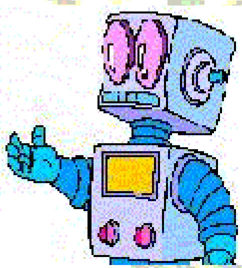
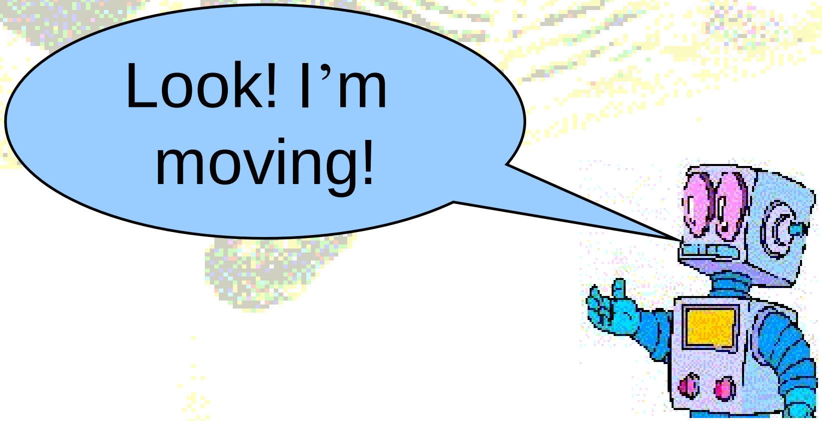
Joint Velocity/Static Forces and the *Jacobian*





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Jacobians: Velocities & Static Forces

- **Jacobians in Robotics:** **Relates joint velocities to Cartesian velocities of the tip of the manipulator arm.**
- **In Mathematics = Multidimensional Derivative**
- **Given a vector function of several variables such as;**

$$\left\{\begin{array}{l} y_1 = f_1(x_1, x_2, \ldots, x_6) \\ y_2 = f_2(x_1, x_2, \ldots, x_6) \\ y_3 = f_3(x_1, x_2, \ldots, x_6) \\ y_4 = f_4(x_1, x_2, \ldots, x_6) \\ y_5 = f_5(x_1, x_2, \ldots, x_6) \\ y_6 = f_6(x_1, x_2, \ldots, x_6) \end{array}\right\} \Rightarrow In \quad Vector \quad Form: \qquad Y = F(X)$$

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Jacobians: Velocities & Static Forces

- Using *Chain-Rule*, differentials of y_i as a function of differentials of x_j are expressed as:

$$\begin{cases} \delta y_1 = \dfrac{\partial f_1}{\partial x_1}\delta x_1 + \dfrac{\partial f_1}{\partial x_2}\delta x_2 + \ldots + \dfrac{\partial f_1}{\partial x_6}\delta x_6 \\ \delta y_2 = \dfrac{\partial f_2}{\partial x_1}\delta x_1 + \dfrac{\partial f_2}{\partial x_2}\delta x_2 + \ldots + \dfrac{\partial f_2}{\partial x_6}\delta x_6 \\ \quad\vdots \\ \delta y_6 = \dfrac{\partial f_6}{\partial x_1}\delta x_1 + \dfrac{\partial f_6}{\partial x_2}\delta x_2 + \ldots + \dfrac{\partial f_6}{\partial x_6}\delta x_6 \end{cases}$$

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➢ **Presenting the differentials using vector notation as:**

$$\underbrace{\delta Y}_{(6\times1)\ \text{Vector}} = \underbrace{\frac{\partial F}{\partial X}}_{(6\times6)\ \text{Matrix}} \underbrace{\delta X}_{(6\times1)\ \text{Vector}}$$

❖ **Jacobian of Partial Derivatives** ⇔ $J \equiv \frac{\partial F}{\partial X}$

❖ **If the functions $f_1(X) \ldots f_6(X)$ are non-linear, then the partial derivatives are a function of x_i, therefore:**

$$\delta Y = \frac{\partial F}{\partial X}\delta X = J(X)\delta X$$

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$$\delta Y = \frac{\partial F}{\partial X}\delta X = J(X)\delta X$$

- **Dividing both sides by the differential time element:**

$$\dot{Y} = J(X)\dot{X}$$

- **Jacobians are time varying linear transformations. At any particular instant, X has a certain value, and J(X) is a linear transformation. At each new instant, X has changed and therefore so has the linear transformation.**

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Jacobians: Velocities & Static Forces

- **In Robotics:** **Jacobian relates joint velocities to Cartesian velocities of the tip of the manipulator arm in a linear fashion.**

$$ {}^{0}V = {}^{0}J(\Theta)\dot{\Theta} $$

- **Where:**

Vector of joint angles: $\Theta = \{\theta_1, \theta_2, \ldots\}$

Vector of joint rates: $\dot{\Theta} = \{\dot{\theta}_1, \dot{\theta}_2, \ldots\}$

Jacobian expressed in frame {0}: ${}^{0}J(\Theta)$

Vector of Cartesian tip velocities in frame {0}: ${}^{0}V$

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Jacobians: Velocities & Static Forces

- **Note that this is an instantaneous relationship, since in the next instant the Jacobian has changed slightly.**

- **For a robot with 6-joints:**

- **Jacobian is a (6×6) matrix:** $^{0}J(\Theta)$
- **Vector of joint rates is a (6×1) vector:** $\dot{\Theta} = \{\dot{\theta}_1, \dot{\theta}_2, \ldots\}$
- **Vector of Cartesian tip velocity is a (6×1) vector:**

$$^{0}V = \begin{bmatrix} ^{0}v_{(3\times 1)} \\ ^{0}\omega_{(3\times 1)} \end{bmatrix}$$

Linear Velocity Vector

Rotational Velocity Vector

Jacobian in general is an (m × n) matrix = $J_{m\times n}$ **:**
# of Rows = # of D.O.F. in Cartesian Space = m
# of Columns = # of Joints of the Manipulator = n

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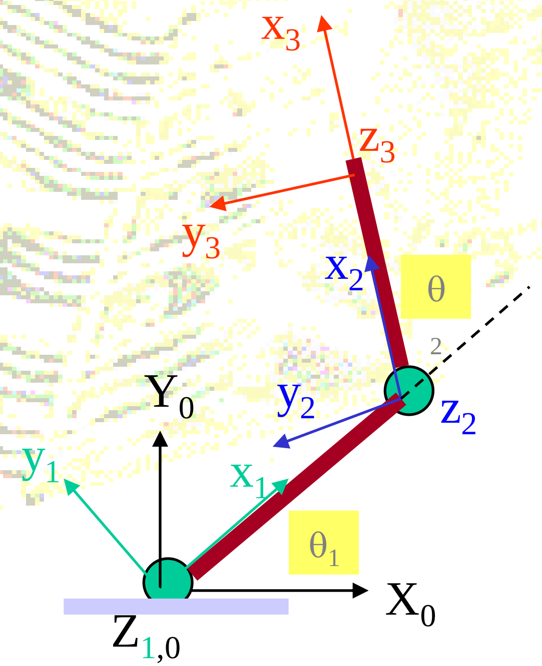
Jacobians: Velocities & Static Forces

- **Example:** **Consider the 2-link manipulator shown. Tip velocities as a function of joint rates $(\dot{\theta}_1, \dot{\theta}_2)$ in terms of frames {0} and {3} are:**



$$
{}^{3}v_3 = \begin{bmatrix} \ell_1 S_2 \dot{\theta}_1 \\ \ell_1 C_2 \dot{\theta}_1 + \ell_2(\dot{\theta}_1 + \dot{\theta}_2) \end{bmatrix} = \underbrace{\begin{bmatrix} \ell_1 S_2 & 0 \\ \ell_1 C_2 + \ell_2 & \ell_2 \end{bmatrix}}_{{}^{3}J(\Theta)} \cdot \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}
$$

$$
{}^{0}v_3 = {}^{0}_{3}R \, {}^{3}v_3 = \begin{bmatrix} -\ell_1 S_1 \dot{\theta}_1 - \ell_2 S_{12}(\dot{\theta}_1 + \dot{\theta}_2) \\ \ell_1 C_1 \dot{\theta}_1 + \ell_2 C_{12}(\dot{\theta}_1 + \dot{\theta}_2) \end{bmatrix} = \underbrace{\begin{bmatrix} -\ell_1 S_1 - \ell_2 S_{12} & -\ell_2 S_{12} \\ \ell_1 C_1 + \ell_2 C_{12} & \ell_2 C_{12} \end{bmatrix}}_{{}^{0}J(\Theta)} \cdot \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}
$$

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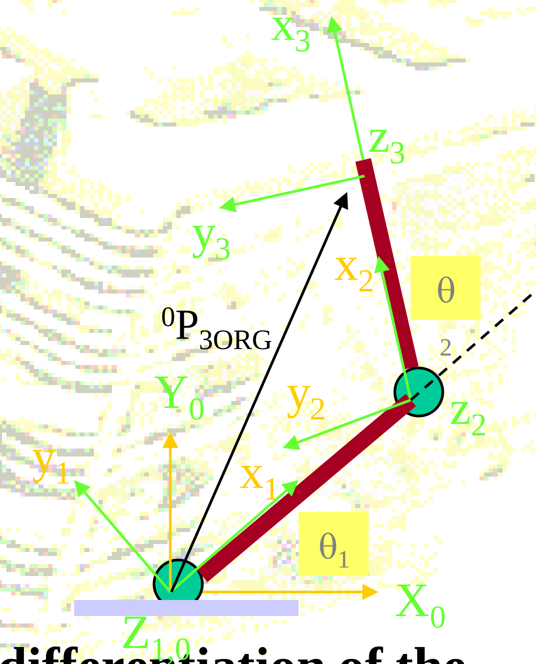
Jacobians: Velocities & Static Forces

- Considering both linear and angular velocity of the end-effector, we have:

$$\begin{bmatrix} {}^3v_3 \\ {}^3\omega_3 \end{bmatrix} = \begin{bmatrix} \ell_1 S_2 \dot{\theta}_1 \\ \ell_1 C_2 \dot{\theta}_1 + \ell_2(\dot{\theta}_1 + \dot{\theta}_2) \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \ell_1 S_2 & 0 \\ \ell_1 C_2 + \ell_2 & \ell_2 \\ 1 & 1 \end{bmatrix}}_{{}^3J(\Theta)_{(3\times 2)}} \cdot \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$



- **Another method to compute Jacobian is by direct differentiation of the kinematics equations. Let X and Y be the two components of ${}^0P_{3ORG}$ vector:**

$$\begin{cases} X = \ell_1 C_1 + \ell_2 C_{12} = f_1(\theta_1, \theta_2) \\ Y = \ell_1 S_1 + \ell_2 S_{12} = f_2(\theta_1, \theta_2) \end{cases} \Rightarrow$$

$${}^0J(\Theta) = \begin{bmatrix} \dfrac{\partial X}{\partial \theta_1} & \dfrac{\partial X}{\partial \theta_2} \\ \dfrac{\partial Y}{\partial \theta_1} & \dfrac{\partial Y}{\partial \theta_2} \end{bmatrix} = \begin{bmatrix} -\ell_1 S_1 - \ell_2 S_{12} & -\ell_2 S_{12} \\ \ell_1 C_1 + \ell_2 C_{12} & \ell_2 C_{12} \end{bmatrix}$$

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Jacobians: Velocities & Static Forces

- Note that Jacobian may also be expressed in another frame by:

$$\boxed{{}^{0}J(\Theta)={}^{0}_{3}R\,{}^{3}J(\Theta)} \longleftrightarrow \boxed{{}^{3}J(\Theta)={}^{0}_{3}R^{-1}\,{}^{0}J(\Theta)}$$

- In general, given a Jacobian written in frame **{B}**, like:

$$\boxed{\begin{bmatrix} {}^{B}v \\ {}^{B}\omega \end{bmatrix}={}^{B}J(\Theta)\dot{\Theta}}$$

- We may like to express the Jacobian in another frame **{A}**. Note that the (6×1) Cartesian velocity vector given in frame {B} can be described in frame {A} by the transformation:

$$\begin{bmatrix} {}^{A}v \\ {}^{A}\omega \end{bmatrix}=\begin{bmatrix} {}^{A}_{B}R & 0 \\ 0 & {}^{A}_{B}R \end{bmatrix}\begin{bmatrix} {}^{B}v \\ {}^{B}\omega \end{bmatrix}=\begin{bmatrix} {}^{A}_{B}R & 0 \\ 0 & {}^{A}_{B}R \end{bmatrix}{}^{B}J(\Theta)\dot{\Theta} \Rightarrow$$

$${}^{A}J(\Theta)=\begin{bmatrix} {}^{A}_{B}R & 0 \\ 0 & {}^{A}_{B}R \end{bmatrix}{}^{B}J(\Theta)$$

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Jacobians: Velocities & Static Forces

- **Singularities(نقاط تکین): We defined ;** $V = J(\Theta)\dot{\Theta}$
- **How about if we need to compute:** $\dot{\Theta} = J^{-1}(\Theta)V$
- **Is J invertible?**

- **Ex: We wish the robot hand to move with a certain velocity in Cartesian space. What would be the necessary joint rates at each instant along the path?**

- **J is singular when |J|=0 (No Inverse for J).**

- **Singularities exits:**
  - **At the boundaries of the workspace (Boundary Singularities),**
  - **When two or more axis line-up (Interior Singularities).**

- **Note that at Singular positions, manipulators lose one or more degrees of freedom.**

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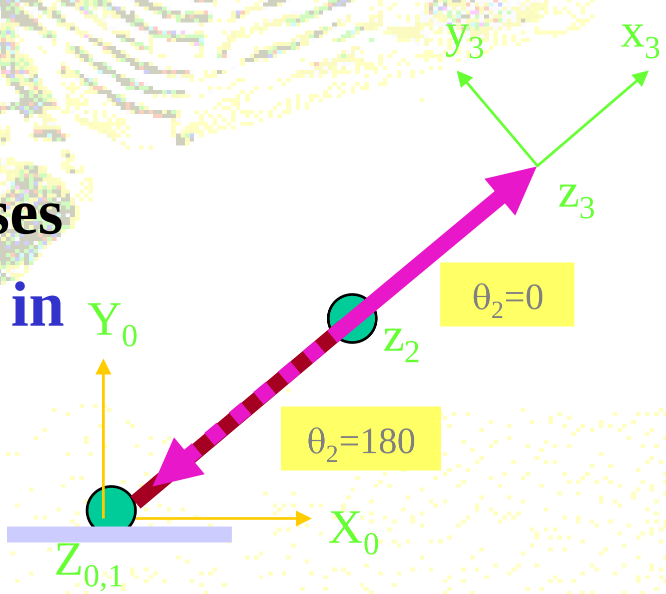
Jacobians: Velocities & Static Forces

- **Example:**

$$
{}^{3}J = \begin{vmatrix} \ell_1 S_2 & 0 \\ \ell_1 C_2 + \ell_2 & \ell_2 \end{vmatrix} = \ell_1 \ell_2 S_2 = 0 \Rightarrow S_2 = 0 \Rightarrow \theta_2 = 0, \pi
$$

- **${}^{0}J$ = {Gives the same results}.**

- **At these positions, manipulator loses one degree of freedom (No motion in the x_3-direction).**



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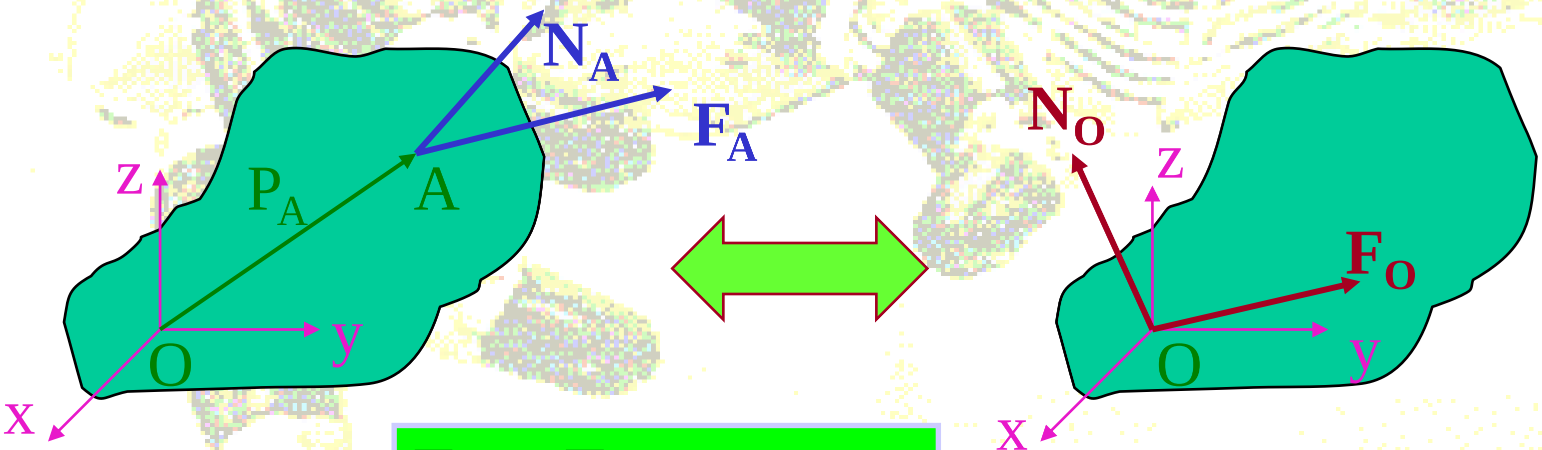
Jacobians: Velocities & Static Forces

- **Static Forces in Manipulators:** Given a desired contact force and moment, what set of joint torques are required to generate them? Jacobian relates joint torques to Cartesian forces of the tip of the manipulator arm in a linear fashion.

- **Recall:** A set of forces and moment acting on a body may be combined into a single force and a single moment at a point.



$$F_O = F_A$$

$$N_O = N_A + P_A \times F_A$$

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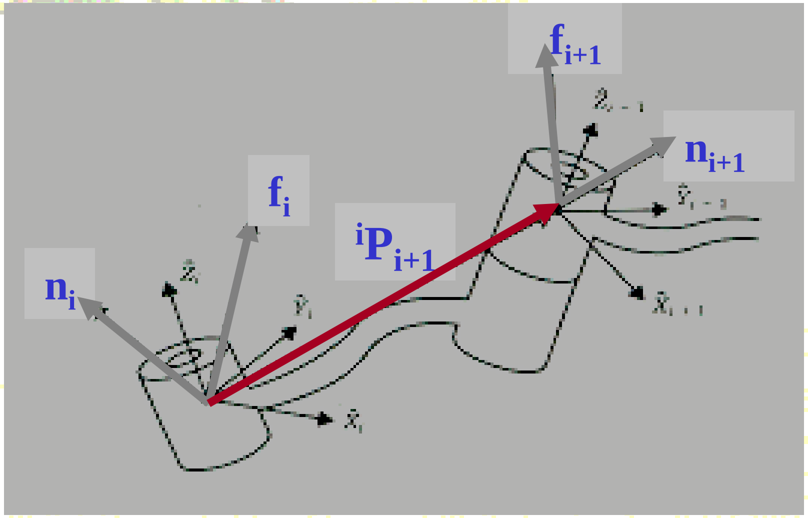
Jacobians: Velocities & Static Forces

➢ To Consider Static Forces in Manipulators:

- **First lock all joints so that the manipulator becomes a structure,**
- **Then write the force-moment balance for each link in terms of the link frames,**
- **Finally compute the static torque at the joints for the manipulator to be in static equilibrium.**

❖ **Force Propagation from Link to Link: Start from the robot hand back to the base (force application at hand is known).**

- **f_i : force exerted on link i by link $i-1$,**
- **n_i : torque exerted on link i by link $i-1$,**
- **f_{i+1} : force exerted on link $i+1$ by link i,**
- **n_{i+1} : torque exerted on link $i+1$ by link i.**



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Jacobians: Velocities & Static Forces

- Express forces and moments in frame {i} as:

$$ {}^{i}f_i = {}^{i}f_{i+1} $$
$$ {}^{i}n_i = {}^{i}n_{i+1} + {}^{i}P_{i+1} \times {}^{i}f_{i+1} $$

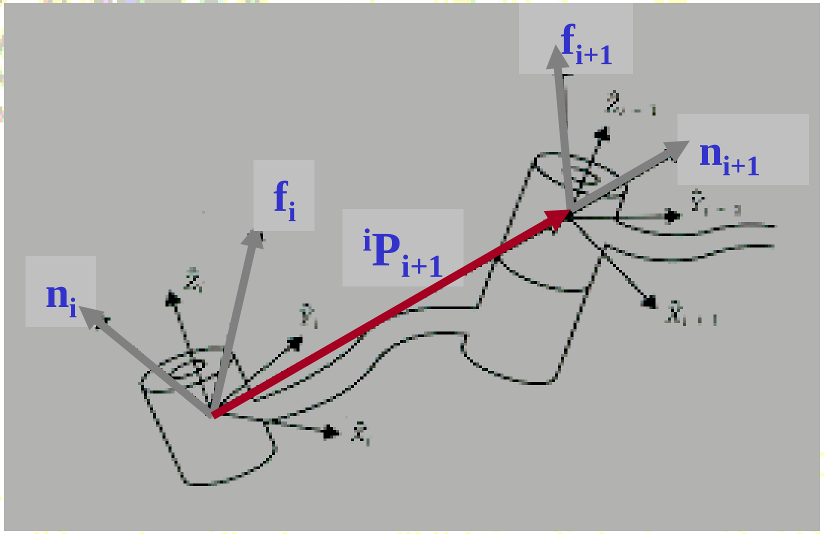
- Now use the rotation matrix ${}^{i}_{i+1}R$ to relate ${}^{i+1}f_{i+1}$ to the frame {i} as:

$$ {}^{i}f_i = {}^{i}_{i+1}R\,{}^{i+1}f_{i+1} = {}^{i}f_{i+1} $$
$$ {}^{i}n_i = {}^{i}_{i+1}R\,{}^{i+1}n_{i+1} + {}^{i}P_{i+1} \times {}^{i}f_i $$

5* An Important Relation

$$ \tau_i = {}^{i}n_i^{T}\,\hat{Z}_i $$

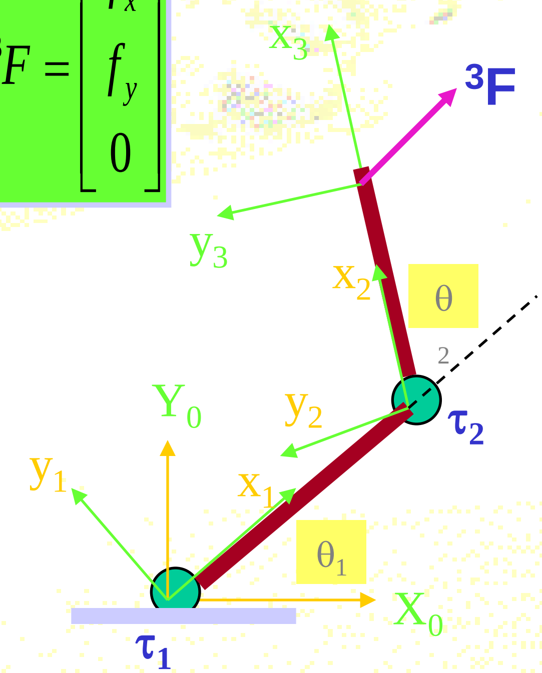
(The z-component of n, useful for control.)



Jacobians: Velocities & Static Forces

- **Example:** **Consider the 2-link arm shown. Gripper is applying a force vector ^3F at the origin of frame {3}. Compute the required joint torques as a function of configuration of the applied force ($\tau_1, \tau_2 = ?$)**

$${}^3f_3 = {}^3F = \begin{bmatrix} f_x \\ f_y \\ 0 \end{bmatrix}$$



- **From equation 5* we have:**

$${}^2f_2 = {}_3^2R\,{}^3f_3 = I\,{}^3f_3 = \begin{bmatrix} f_x \\ f_y \\ 0 \end{bmatrix}$$

$${}^2n_2 = {}_3^2R\,\underbrace{{}^3n_3}_{0} + {}^2P_3 \times {}^2f_2 = (\ell_2 \hat{X}_2) \times \begin{bmatrix} f_x \\ f_y \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ \ell_2 f_y \end{bmatrix}}_{\tau_2}$$

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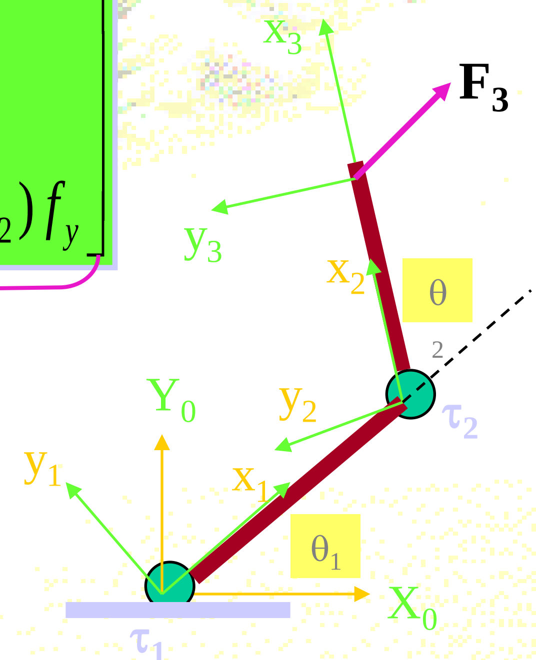
Jacobians: Velocities & Static Forces

❖ **Reapplying equation 5* we have:**

$$^{1}f_1 = {}^{1}_{2}R\,{}^{2}f_2 = \begin{bmatrix} C_2 & -S_2 & 0 \\ S_2 & C_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} f_x \\ f_y \\ 0 \end{bmatrix} = \begin{bmatrix} C_2 f_x - S_2 f_y \\ S_2 f_x + C_2 f_y \\ 0 \end{bmatrix}$$

$$^{3}f_3 = {}^{3}F = \begin{bmatrix} f_x \\ f_y \\ 0 \end{bmatrix}$$

$$^{1}n_1 = {}^{1}_{2}R\,{}^{2}n_2 + {}^{1}P_2 \times {}^{1}f_1 = \begin{bmatrix} 0 \\ 0 \\ \ell_2 f_y \end{bmatrix} + (\ell_1 \hat{X}_1) \times {}^{1}f_1 = \begin{bmatrix} 0 \\ 0 \\ \underbrace{\ell_2 S_2 f_x + (\ell_1 C_2 + \ell_2) f_y}_{\tau_1} \end{bmatrix}$$



$$\tau = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} \ell_1 S_2 & \ell_2 + \ell_1 C_2 \\ 0 & \ell_2 \end{bmatrix}\begin{bmatrix} f_x \\ f_y \end{bmatrix} = \left[{}^{3}J\right]^{T}\,{}^{3}F$$

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Jacobians: Velocities & Static Forces

- **Jacobians in the Force Domain:**

(هنگامیکه نیرویی بر یک مکانیزم وارد شود، چنانچه مکانیزم تغییر مکان یابد کار انجام میگردد)

Equate the virtual work done by forces in the Cartesian space with that done by joint torques in the joint space. They must be equal unless something is moving. **(Since work has units of energy, it must be the same measured in any set of generalized coordinated)**

$$F \cdot \delta X = \tau \cdot \delta\Theta \Leftrightarrow (Force \cdot Virtual - Displacement)$$

$$F^T \delta X = \tau^T \delta\Theta, \qquad but : \delta X = J \delta\Theta$$

$$\therefore \quad F^T J \cancel{\delta\Theta} = \tau^T \cancel{\delta\Theta} \Rightarrow \tau^T = F^T J \Rightarrow$$

$$\tau = J^T F \qquad , \qquad \tau = {}^0J^T \, {}^0F$$

- **Note that when Jacobian loses full rank (becomes singular), there are certain directions in which the end-effector cannot exert static forces as desired.**

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Exercises:

5.1, 5.3, 5.7, 5.10, 5.12

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