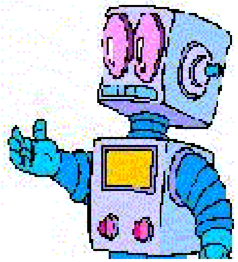
INTRODUCTION TO ROBOTICS

(Kinematics, Dynamics, and Design)

SESSION # 14:

MANIPULATOR'S JACOBIANS



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Joint Velocity/Static Forces and the *Jacobian*

Chapter Objectives

By the end of the Chapter, you should be able to:

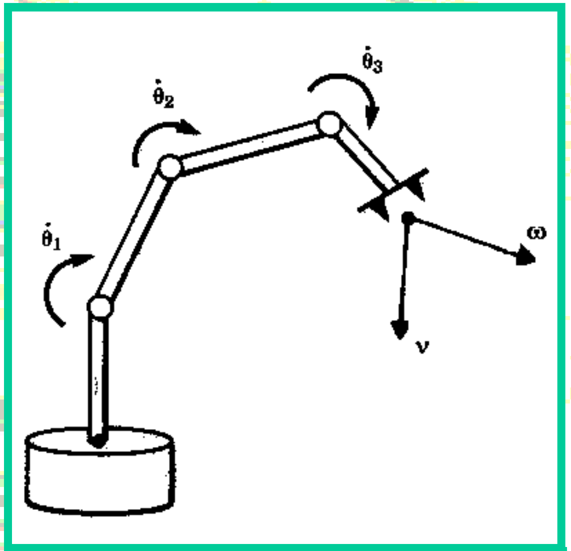
- Characterize frame velocity
- Compute linear and rotational velocity
- Compute Jacobian and robot singularities
- Relate joint forces (forces & torques) to Cartesian forces of the tip of the manipulator arm in a linear fashion

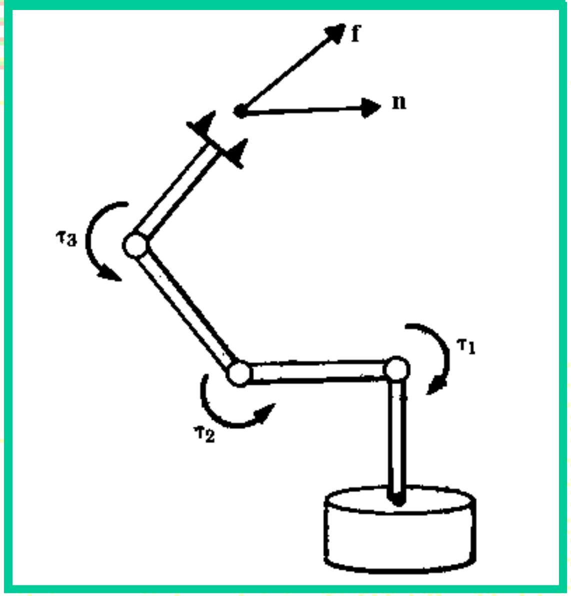
Jacobians: Velocities & Static Forces

➢ **Jacobian of the Manipulator:**

✓ **A matrix quantity called the <u>Jacobian</u> specifies a mapping from velocities in {*Joint Space*} to velocities in {*Cartesian space*}.**

✓ **For a desired contact "static" {force and moment}, Jacobian can also be used to compute the set of {Joint Torques} required to generate them.**





Jacobians: Velocities & Static Forces

- **Studying Dynamics requires knowledge of Velocities and Accelerations:**

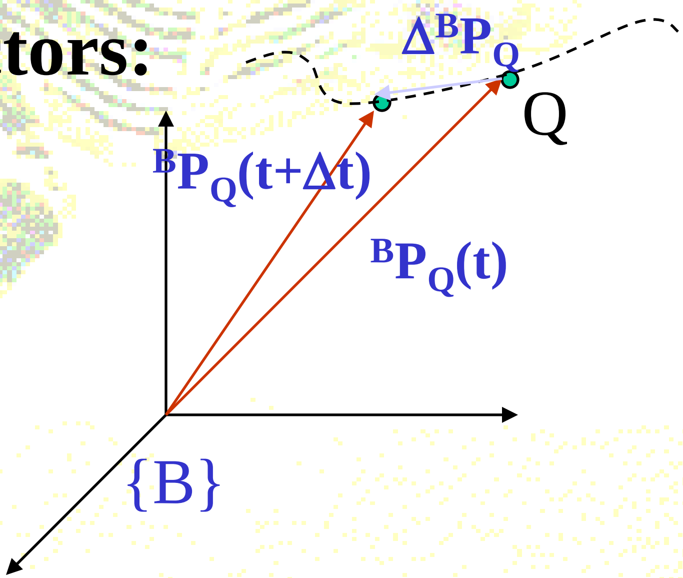
Notation for Time-Varying Position & Orientation:

Differentiation of Position Vectors:

Consider a point Q in space, and the position vector $^{B}P_{Q}$:

$$\Delta {}^{B}P_{Q} = {}^{B}P_{Q}(t+\Delta t) - {}^{B}P_{Q}(t)$$

$${}^{B}V_{Q} = \lim_{\Delta t \to 0} \frac{\Delta {}^{B}P_{Q}}{\Delta t} = \frac{d}{dt}{}^{B}P_{Q}$$

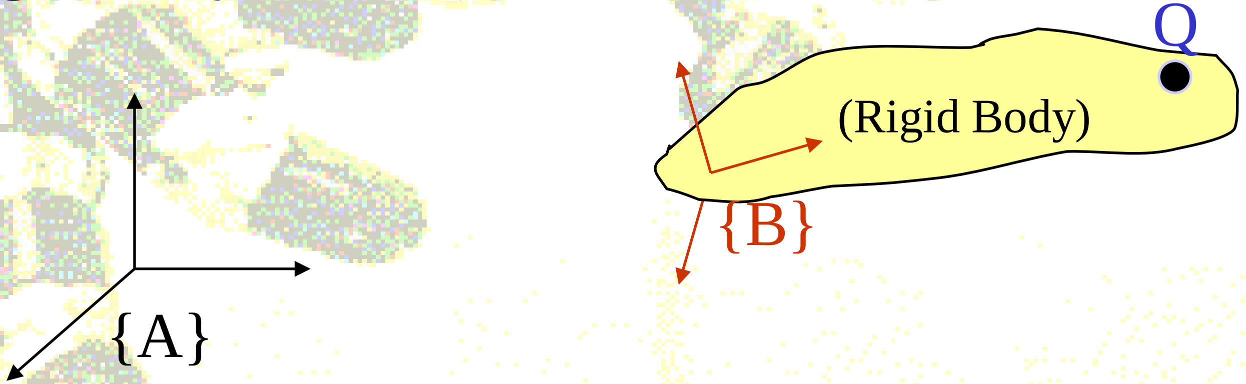


Jacobians: Velocities & Static Forces

- **Velocity of a position vector is the velocity of the point that vector describes:**

If the point Q does not move relative to {B}, then its velocity is zero, even if it moves with respect to another frame like {A}. It is important to indicate the frame in which the position vector is differentiated.

Ex: (Rigid Body Motion)



Jacobians: Velocities & Static Forces

- **Just like any other vector, the Velocity vector can also be described in terms of any frame:**

Ex: The velocity vector "$^{B}V_{Q}$" expressed in terms of another frame like {A}, would be written as:

$$^{A}(^{B}V_{Q}) \equiv {}^{A}\left(\frac{d}{dt}{}^{B}P_{Q}\right) \equiv {}^{A}_{B}R\,{}^{B}V_{Q}$$

Rotation transformation is used to map velocity vector from frame {A} to frame {B}. **(Recall that velocity and accelerations are free vectors)**

(Note that the frame with respect to which the differentiation is done, is important.)

Jacobians: Velocities & Static Forces

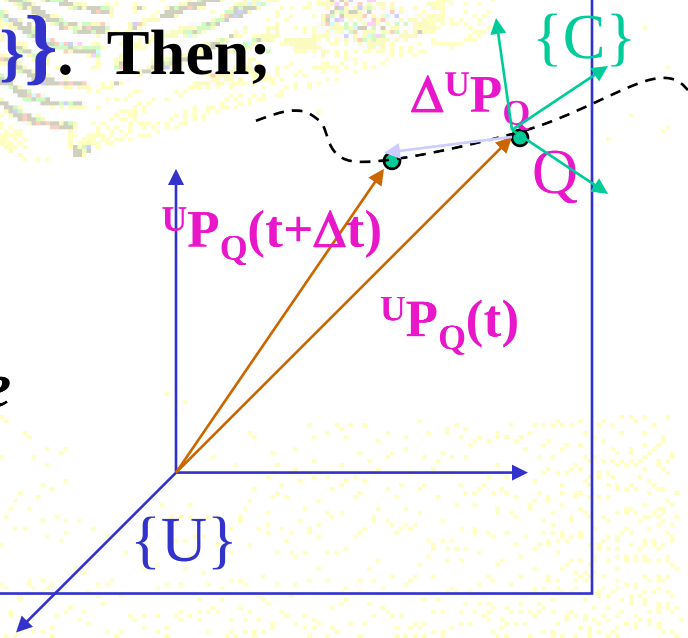
- If the point in question "Q" is the origin of a frame {C}, and the differentiation is done with respect to a *Universe* frame {U}, then we may write:

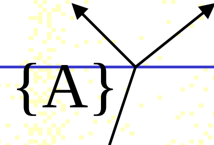
$$v_C \equiv \left(\frac{d}{dt}{}^U P_Q\right) \equiv \left(\frac{d}{dt}{}^U P_{CORG}\right) \equiv {}^U V_{CORG}$$

{velocity of origin of {C} relative to {U}}. Then;

$${}^A(v_C) \equiv {}^A({}^U V_{CORG}) \equiv {}^A v_C$$

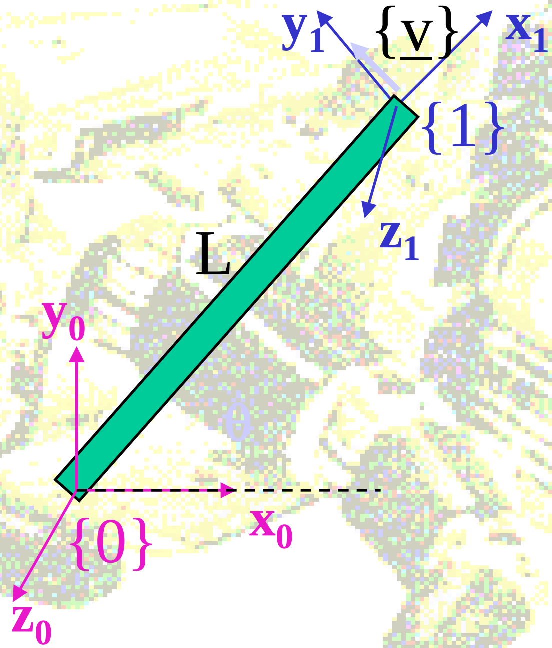
{velocity of $\{C\}_{ORG}$ relative to *Universe* {U}, expressed in frame {A}}.





Jacobians: Velocities & Static Forces

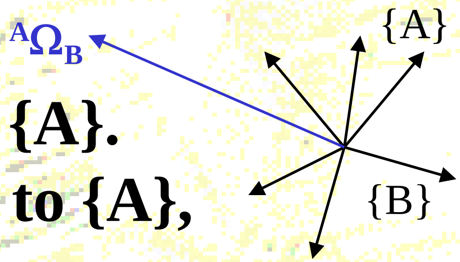
- **Ex:** **Consider the following one-link manipulator as shown:**



$$^{1}({}^{0}V_{1ORG}) \equiv {}^{1}_{0}R\,{}^{0}V_{1ORG} \equiv {}^{0}_{1}R^{-1}\,{}^{0}V_{1ORG} \equiv {}^{1}v_{1ORG} = \begin{Bmatrix} 0 \\ \ell\dot{\theta} \\ 0 \end{Bmatrix}$$

Jacobians: Velocities & Static Forces

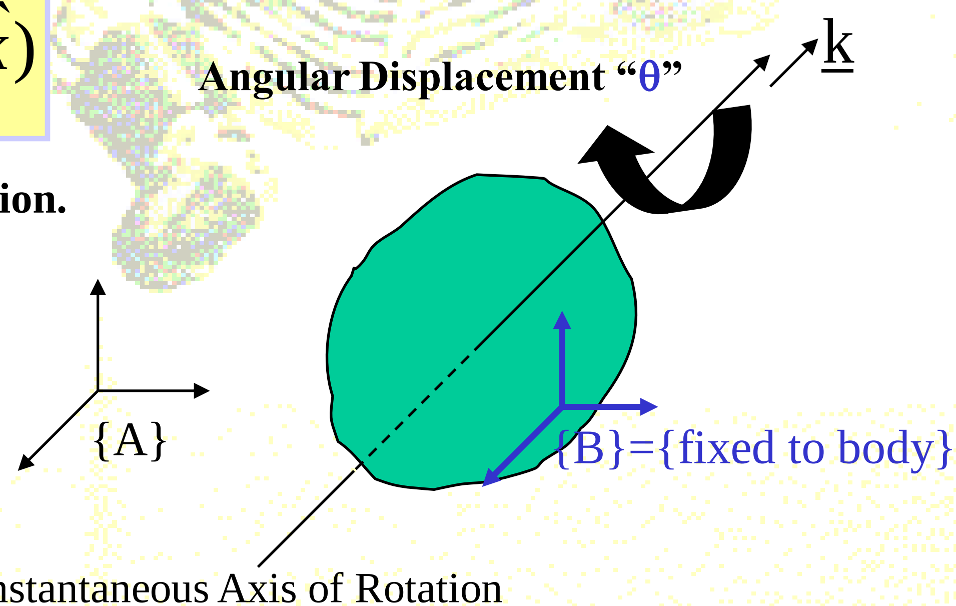
- **The Angular Velocity Vector: Angular velocity "Ω" describes rotational motion of a frame attached to a body. Lets define the following:**



- **$^{A}\Omega_{B}$: Angular Velocity of frame {B} relative to {A}.**
- **$^{C}(^{A}\Omega_{B})$: Angular Velocity of frame {B} relative to {A}, expressed in {C}.**

$$^{A}\Omega_{B} \equiv {}^{A}\left(\frac{d\theta}{dt}\hat{k}\right) \equiv {}^{A}\left(\frac{d\theta}{dt}{}^{B}\hat{k}\right)$$

k: Unit vector along the axis of rotation.



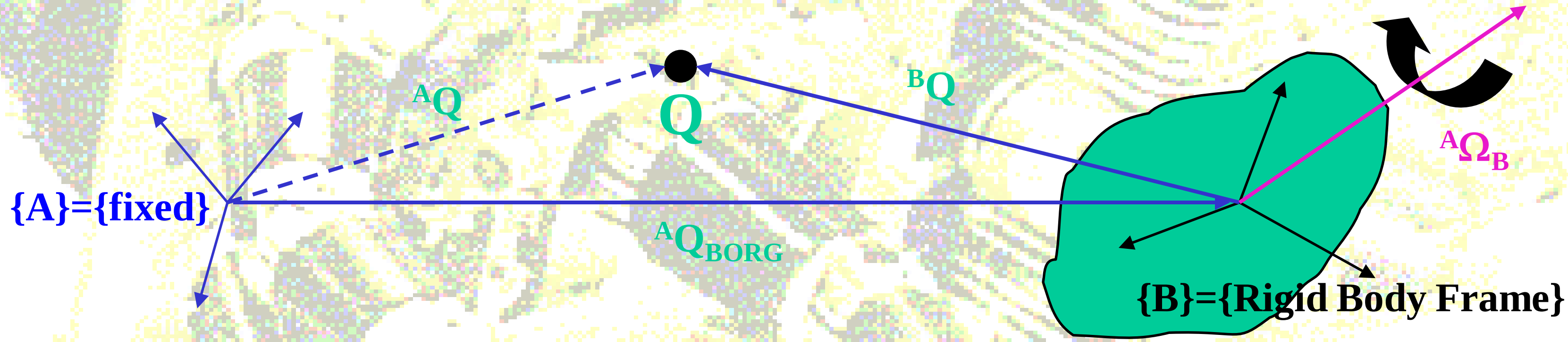
- **Relative to Universal frame, we can write:**

$$\omega_{C} \equiv {}^{U}\Omega_{C}$$
$$^{A}\omega_{C} \equiv {}^{A}({}^{U}\Omega_{C})$$

Jacobians: Velocities & Static Forces

- **Linear and Rotational Velocity of Rigid Bodies:**

Consider a point "Q" in space, and describe its kinematics in two frames {A} and {B}.



From Chapter-2 we have:

$$ {}^{A}Q \equiv {}^{A}Q_{BORG} + {}^{A}_{B}R\,{}^{B}Q $$

- **Differentiating with respect to time results:**

$$ {}^{A}V_{Q} \equiv {}^{A}\dot{Q} = {}^{A}\dot{Q}_{BORG} + {}^{A}_{B}\dot{R}\,{}^{B}Q + {}^{A}_{B}R\,{}^{B}\dot{Q} \qquad \Rightarrow $$

$$ {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}Q + {}^{A}_{B}R\,{}^{B}V_{Q} $$

Jacobians: Velocities & Static Forces

$$ {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}Q + {}^{A}_{B}R\,{}^{B}V_{Q} $$

- **Linear Velocity (Translation Only) of Rigid Body ($^{A}\Omega_{B}=0$):**

$$ {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} $$

- **Angular Velocity (Rotation Only) of Rigid Body ($^{A}V_{BORG}=0$): (Frames {A} and {B} coincident);**

$$ {}^{A}V_{Q} = {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}Q + {}^{A}_{B}R\,{}^{B}V_{Q} $$

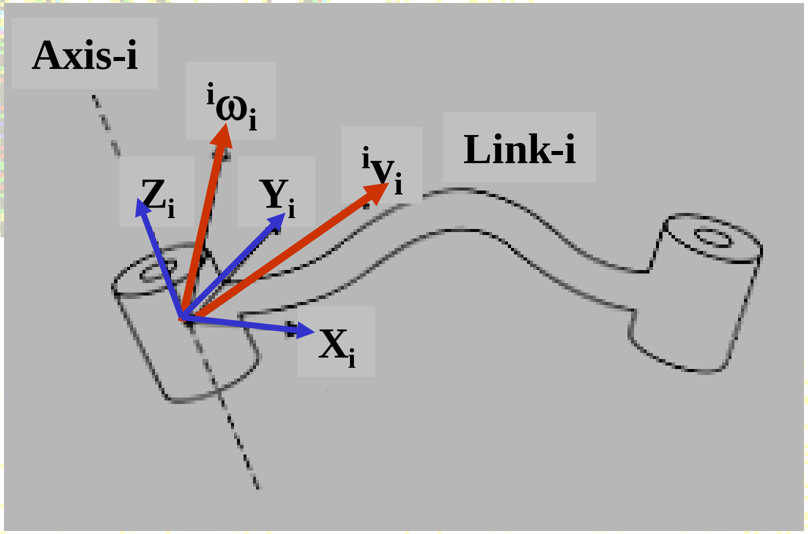
Jacobians: Velocities & Static Forces

- **Motion of the Links of a Robot:** **in studying robot motion, we define:**

- **Frame {0}:** **A reference frame**
- **v_i: Linear velocity of the origin of link frame {i},**
- **ω_i: Angular velocity of the link frame {i}.**

At any instant, each link of a robot in motion has some linear and angular velocity defined by:

- **${}^{i}v_i$: Linear velocity of the origin of link frame {i} with respect to {U}, and written in frame {i},**
- **${}^{i}\omega_i$: Angular velocity of the link frame {i} with respect to {U}, and written in frame{i}.**



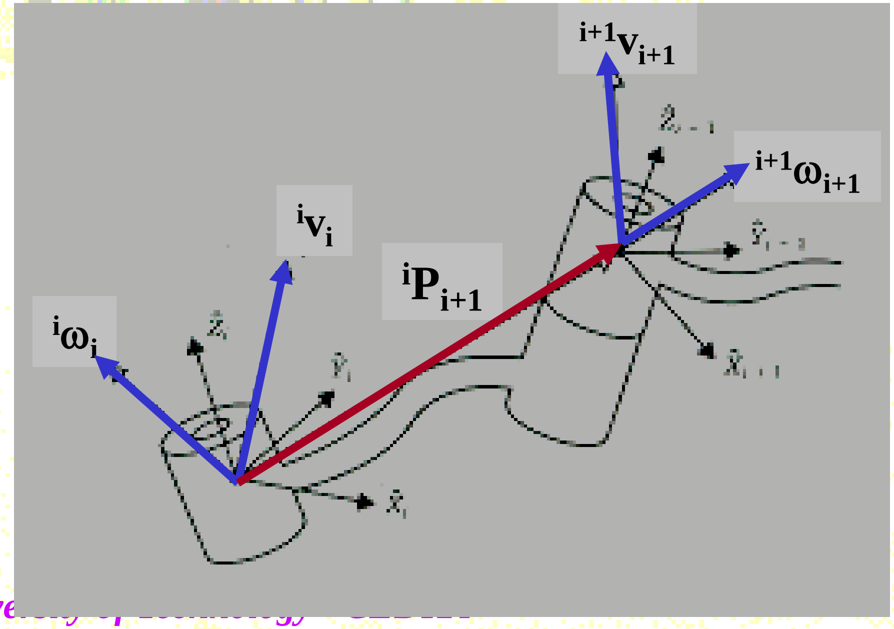
Jacobians: Velocities & Static Forces

- **Velocity Propagation from Link to Link: A manipulator is a chain of rigid bodies, each one capable of motion relative to its neighbors. To study its motion:**

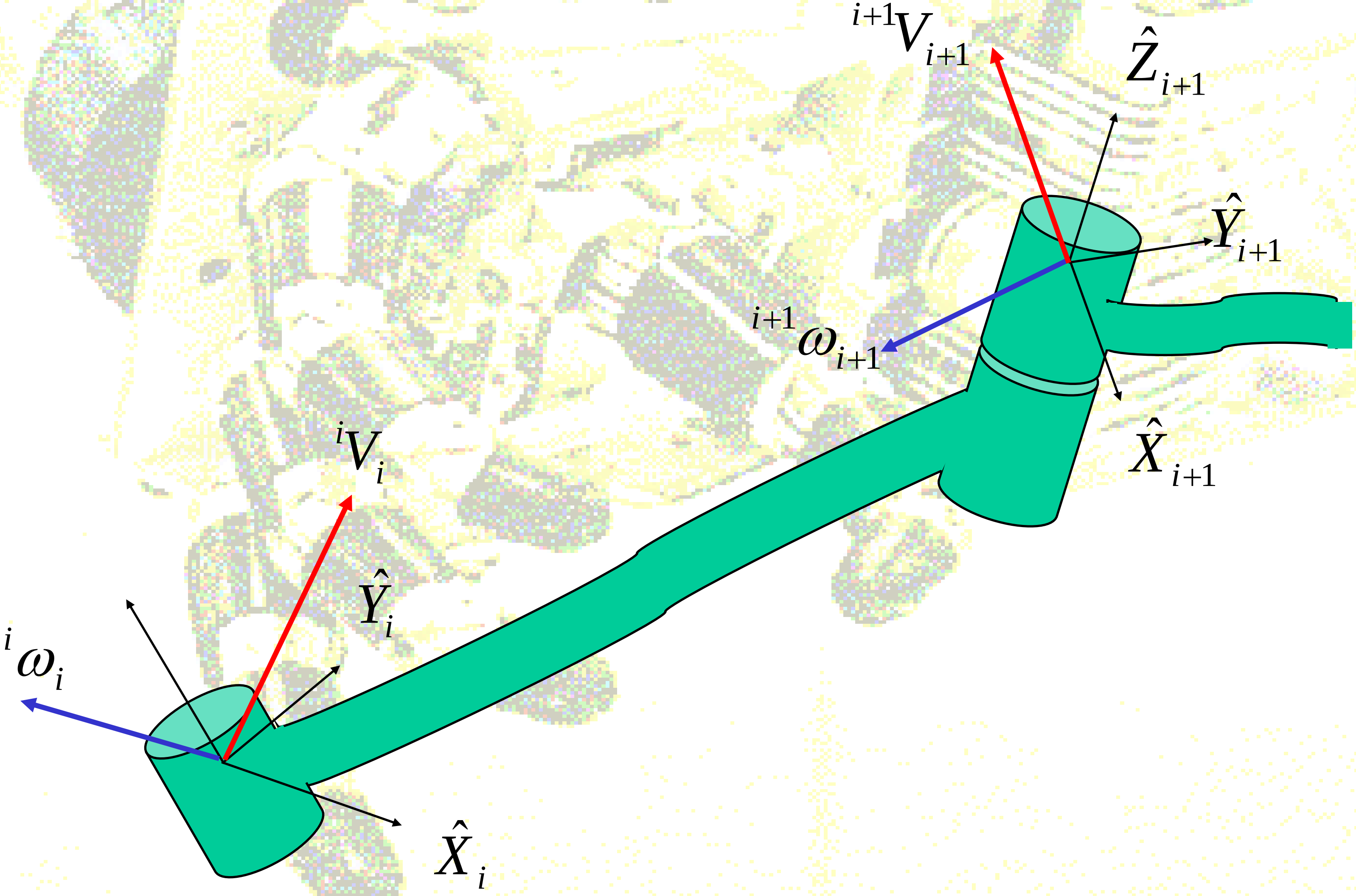
- **Start from base, and work out to link n.**
- **Each link is a R.B. with some v and ω expressed in the link's frame.**
- **Angular velocities from link to link may be added as long as they are expressed in the same frame.**

$$ {}^{i}\omega_{i+1} = {}^{i}\omega_{i} + {}^{i}_{i+1}R\,\dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} $$

$$ {}^{i+1}\begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{bmatrix} $$



Velocity Propagation



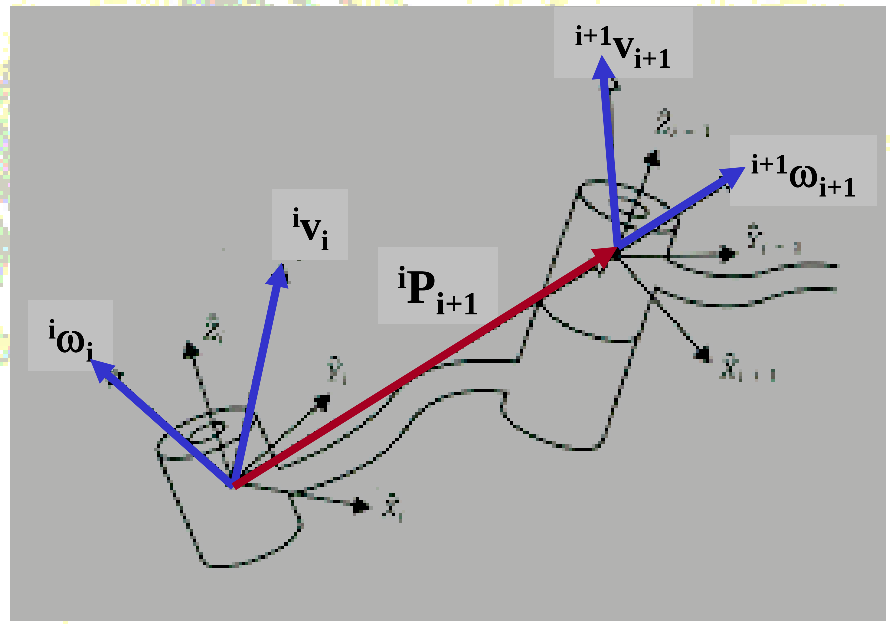
Jacobians: Velocities & Static Forces

➢ Velocity Propagation from Link to Link:

❖ Angular velocity of link i+1 is equal to the angular velocity of link i plus the new angular velocity component at joint i+1, all expressed in frame {i}.

$$ {}^{i}\omega_{i+1} = {}^{i}\omega_{i} + {}^{i}_{i+1}R\,\dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} $$

$$ {}^{i+1}\begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{bmatrix} $$



❖ The R-matrix is used to express the new angular velocity at joint i+1 in frame {i}.

Jacobians: Velocities & Static Forces

➢ **Velocity Propagation from Link to Link:**

❖ **Pre-multiplying both sides of this equation by $^{i+1}_{i}R$, we have:**

$$^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\,{}^{i}\omega_{i} + \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}$$

*** An Important Relation**

$$\dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} = {}^{i+1}\begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{bmatrix}$$

❖ **Dynamics has a *Recursive* nature in manipulators. If you know i, you can find i+1.**

Jacobians: Velocities & Static Forces

➢ Velocity Propagation from Link to Link:

❖ **Linear Velocity of the origin of frame {i+1} is equal to the linear velocity of origin of frame {i} plus the new velocity component due to the rotation of link i, all expressed in frame {i}. Similar to:** $v_B = v_A + \omega \times r_{B/A}$

$$ {}^{i}v_{i+1} = {}^{i}v_i + {}^{i}\omega_i \times {}^{i}P_{i+1} $$

❖ **Pre-multiplying both sides of this equation by** ${}^{i+1}_{i}R$ **, we have:**

$$ {}^{i+1}v_{i+1} = {}^{i+1}_{i}R({}^{i}v_i + {}^{i}\omega_i \times {}^{i}P_{i+1}) $$

**** An Important Relation**

❖ **Equations (*) and (**) are for when the joint i+1 is *Revolute*.**

Jacobians: Velocities & Static Forces

➢ Velocity Propagation from Link to Link:

- If the **joint i+1** is ***Prismatic (Sliding)***, then we have:

$$^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\,{}^{i}\omega_{i}$$

3* An Important Relation

$$^{i+1}v_{i+1} = {}^{i+1}_{i}R\left({}^{i}v_{i} + {}^{i}\omega_{i} \times {}^{i}P_{i+1}\right) + \dot{d}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}$$

4* An Important Relation

- Using these relations from link to link one can compute the linear "$^{n}v_{n}$" and angular "$^{n}\omega_{n}$" velocities of the last link of the manipulator.
- If we wish to compute the linear and angular velocities of the **last link n** in terms of frame **{0}**, we can compute them as follows:

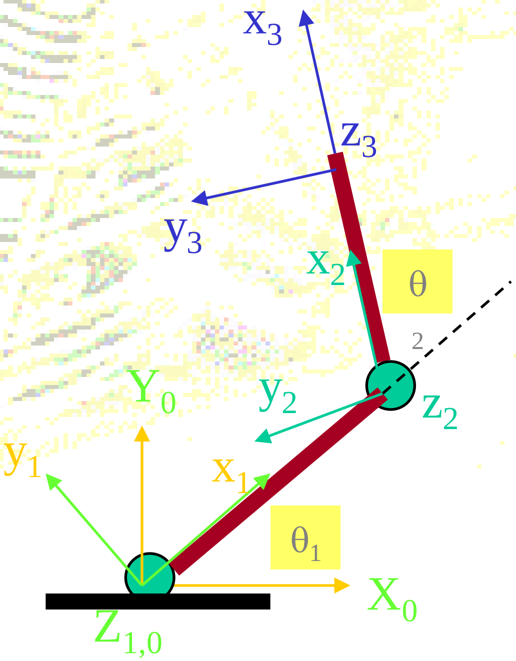
$$^{0}\omega_{n} = {}^{0}_{n}R\,{}^{n}\omega_{n}, \quad and \quad {}^{0}v_{n} = {}^{0}_{n}R\,{}^{n}v_{n}$$

Jacobians: Velocities & Static Forces

- **Example:** **Consider the 2-link manipulator shown. Find the tip velocity as a function of joint rates $(\dot{\theta}_1, \dot{\theta}_2)$ in terms of frames {0} and {3}?**

- **Since the joints are *Revolute*, then:**

$$\begin{cases} {}^{i+1}\omega_{i+1} = {}^{i+1}_{\ \ i}R\,{}^{i}\omega_i + \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} \\ {}^{i+1}v_{i+1} = {}^{i+1}_{\ \ i}R\left({}^{i}v_i + {}^{i}\omega_i \times {}^{i}P_{i+1}\right) \end{cases}$$



$${}^{0}_{1}T = \begin{bmatrix} C_1 & -S_1 & 0 & 0 \\ S_1 & C_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^{1}_{2}T = \begin{bmatrix} C_2 & -S_2 & 0 & \ell_1 \\ S_2 & C_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^{2}_{3}T = \begin{bmatrix} 1 & 0 & 0 & \ell_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Jacobians: Velocities & Static Forces

- Start from the fixed frame {0}, or i=0:

$$^{1}\omega_1 = {}^{1}_{0}R\,{}^{0}\omega_0 + \dot{\theta}_1\,{}^{1}\hat{Z}_1 = \dot{\theta}_1\,{}^{1}\hat{Z}_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix}, \qquad {}^{1}v_1 = {}^{1}_{0}R({}^{0}v_0 + {}^{0}\omega_0 \times {}^{0}P_1) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

($^{0}\omega_0 = 0$, $^{0}v_0 = 0$, $^{0}\omega_0 = 0$)

- For i=1:

$$^{2}\omega_2 = {}^{2}_{1}R\,{}^{1}\omega_1 + \dot{\theta}_2\,{}^{2}\hat{Z}_2 = \dot{\theta}\begin{bmatrix} C_2 & S_2 & 0 \\ -S_2 & C_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix}$$

$$^{2}v_2 = {}^{2}_{1}R({}^{1}v_1 + {}^{1}\omega_1 \times {}^{1}P_2) = {}^{2}_{1}R(\ell_1\dot{\theta}_1\,{}^{1}\hat{Y}_1) = {}^{2}_{1}R\begin{bmatrix} 0 \\ \ell_1\dot{\theta}_1 \\ 0 \end{bmatrix} = \begin{bmatrix} \ell_1 S_2\dot{\theta}_1 \\ \ell_1 C_2\dot{\theta}_1 \\ 0 \end{bmatrix}$$

($^{1}v_1 = 0$)

$$^{1}\omega_1 \times {}^{1}P_2 = (\dot{\theta}_1\,{}^{1}\hat{Z}_1) \times (\ell_1\,{}^{1}\hat{X}_1) = (\ell_1\dot{\theta}_1\,{}^{1}\hat{Y}_1)$$

Jacobians: Velocities & Static Forces

- **For i=2:**

$$^{3}\omega_3 = {}^{3}_{2}R\,{}^{2}\omega_2 + \dot{\theta}_3\,{}^{3}\hat{Z}_3 = {}^{2}\omega_2$$

I 0

$$^{3}v_3 = {}^{3}_{2}R({}^{2}v_2 + {}^{2}\omega_2 \times {}^{2}P_3) = \begin{bmatrix} \ell_1 S_2 \dot{\theta}_1 \\ \ell_1 C_2 \dot{\theta}_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \ell_2(\dot{\theta}_1 + \dot{\theta}_2) \\ 0 \end{bmatrix} = \begin{bmatrix} \ell_1 S_2 \dot{\theta}_1 \\ \ell_1 C_2 \dot{\theta}_1 + \ell_2(\dot{\theta}_1 + \dot{\theta}_2) \\ 0 \end{bmatrix}$$

I

$$^{2}\omega_2 \times {}^{2}P_3 = (\dot{\theta}_1 + \dot{\theta}_2)\,{}^{2}\hat{Z}_2 \times (\ell_2\,{}^{2}\hat{X}_2) = \ell_2(\dot{\theta}_1 + \dot{\theta}_2)\,{}^{2}\hat{Y}_2$$

$$^{0}v_3 = {}^{0}_{3}R\,{}^{3}v_3 = \begin{bmatrix} -\ell_1 S_1 \dot{\theta}_1 - \ell_2 S_{12}(\dot{\theta}_1 + \dot{\theta}_2) \\ \ell_1 C_1 \dot{\theta}_1 + \ell_2 C_{12}(\dot{\theta}_1 + \dot{\theta}_2) \\ 0 \end{bmatrix}$$

$$^{0}_{3}R = \begin{bmatrix} C_{12} & -S_{12} & 0 \\ S_{12} & C_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Jacobians: Velocities & Static Forces

- **Jacobians in Robotics: Relates joint velocities to Cartesian velocities of the tip of the manipulator arm.**
- **In Mathematics = Multidimensional Derivative**
- **Given a vector function of several variables such as;**

$$\begin{cases} y_1 = f_1(x_1, x_2, \ldots, x_6) \\ y_2 = f_2(x_1, x_2, \ldots, x_6) \\ y_3 = f_3(x_1, x_2, \ldots, x_6) \\ y_4 = f_4(x_1, x_2, \ldots, x_6) \\ y_5 = f_5(x_1, x_2, \ldots, x_6) \\ y_6 = f_6(x_1, x_2, \ldots, x_6) \end{cases} \Rightarrow In \quad Vector \quad Form: \qquad Y = F(X)$$

Jacobians: Velocities & Static Forces

- **Using *Chain-Rule*, differentials of y_i as a function of differentials of x_j are expressed as:**

$$\begin{cases} \delta y_1 = \dfrac{\partial f_1}{\partial x_1}\delta x_1 + \dfrac{\partial f_1}{\partial x_2}\delta x_2 + \ldots + \dfrac{\partial f_1}{\partial x_6}\delta x_6 \\ \delta y_2 = \dfrac{\partial f_2}{\partial x_1}\delta x_1 + \dfrac{\partial f_2}{\partial x_2}\delta x_2 + \ldots + \dfrac{\partial f_2}{\partial x_6}\delta x_6 \\ \quad\quad\quad\quad . \\ \quad\quad\quad\quad . \\ \quad\quad\quad\quad . \\ \delta y_6 = \dfrac{\partial f_6}{\partial x_1}\delta x_1 + \dfrac{\partial f_6}{\partial x_2}\delta x_2 + \ldots + \dfrac{\partial f_6}{\partial x_6}\delta x_6 \end{cases}$$

Jacobians: Velocities & Static Forces

- **Presenting the differentials using vector notation as:**

$$\delta Y = \frac{\partial F}{\partial X}\delta X$$

(6×1) Vector **(6×6) Matrix** **(6×1) Vector**

- **Jacobian of Partial Derivatives** ⇔ $J \equiv \frac{\partial F}{\partial X}$

- **If the functions $f_1(X) \ldots f_6(X)$ are non-linear, then the partial derivatives are a function of x_i, therefore:**

$$\delta Y = \frac{\partial F}{\partial X}\delta X = J(X)\delta X$$

Jacobians: Velocities & Static Forces

$$\delta Y = \frac{\partial F}{\partial X}\delta X = J(X)\delta X$$

- **Dividing both sides by the differential time element:**

$$\dot{Y} = J(X)\dot{X}$$

- **Jacobians are time varying linear transformations. At any particular instant, X has a certain value, and J(X) is a linear transformation. At each new instant, X has changed and therefore so has the linear transformation.**

Jacobians: Velocities & Static Forces

- **In Robotics:** **Jacobian relates joint velocities to Cartesian velocities of the tip of the manipulator arm in a linear fashion.**

$$^{0}V = {}^{0}J(\Theta)\dot{\Theta}$$

- **Where:**

Vector of joint angles: $\Theta = \{\theta_1, \theta_2, \ldots\}$

Vector of joint rates: $\dot{\Theta} = \{\dot{\theta}_1, \dot{\theta}_2, \ldots\}$

Jacobian expressed in frame {0}: $^{0}J(\Theta)$

Vector of Cartesian tip velocities in frame {0}: ^{0}V

Jacobians: Velocities & Static Forces

- Note that this is an instantaneous relationship, since in the next instant the Jacobian has changed slightly.

- **For a robot with 6-joints:**

- Jacobian is a (6×6) matrix: $^{0}J(\Theta)$
- Vector of joint rates is a (6×1) vector: $\dot{\Theta} = \{\dot{\theta}_1, \dot{\theta}_2, \ldots\}$
- Vector of Cartesian tip velocity is a (6×1) vector:

$$^{0}V = \begin{bmatrix} {}^{0}v_{(3\times1)} \\ {}^{0}\omega_{(3\times1)} \end{bmatrix}$$

${}^{0}v_{(3\times1)}$ ↔ Linear Velocity Vector

${}^{0}\omega_{(3\times1)}$ ↔ Rotational Velocity Vector

Jacobian in general is an (m × n) matrix = $J_{m\times n}$:
# of Rows = # of D.O.F. in Cartesian Space = m
# of Columns = # of Joints of the Manipulator = n