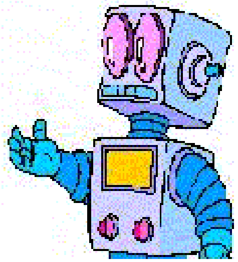
INTRODUCTION TO ROBOTICS

(Kinematics, Dynamics, and Design)

SESSION # 13:

MANIPULATOR

INVERSE KINEMATICS



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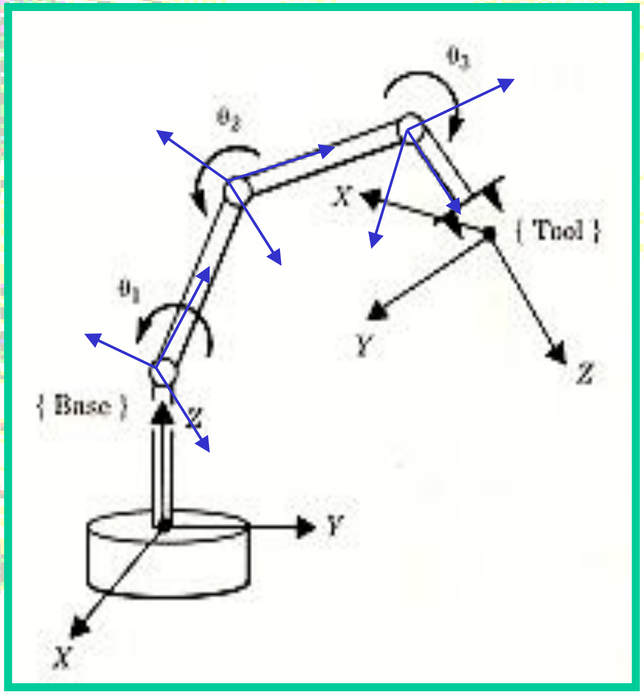
Inverse Manipulator Kinematics

- **Forward Kinematics:** Describe the position and orientation of the manipulator's end-effector as a function joint variables relative to a base frame.

- **Inverse Kinematics:** Given the desired position and orientation of the end-effector relative to the base, compute the set of joint variables which will achieve this desired result.



A 3-DOF Manipulator Arm

Inverse Manipulator Kinematics

- **Solvability (قابل حل بودن):**

  Solving kinematics equations in robotics is a Non-Linear Problem.

  Given; ${}^0_nT$, Find; $\{\theta_1, \theta_2, \ldots, \theta_n\}$, is a *non-linear* problem.

  Ex: PUMA-560 Robot. Given; ${}^0_6T$, Find; $\{\theta_1, \theta_2, \ldots, \theta_6\}$, (see Equation 3.14)

  For a 6-DOF manipulator, we have:

$$
{}^0_6T = \left[\begin{array}{ccc|c} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ \hline 0 & 0 & 0 & 1 \end{array}\right]
$$

- **12-Equations, and 6-Unknowns?**
- **From 9-Equations of the Rotation Matrix, only 3-Equations are independent.**
- **Therefore, we have 6-independent non-linear equations and 6-unknowns.**

Inverse Manipulator Kinematics

- **Solvability (قابل حل بودن):**

  We have 6-independent non-linear equations and 6-unknowns. Therefore, we should investigate the followings:

- ➢ **Existence of Solution (وجود جواب) .**
- ➢ **Multiple Solutions (تعدد جوابها) .**
- ➢ **Method of Solution (روش حل) .**

Inverse Manipulator Kinematics

- **Solvability (قابل حل بودن):**
  - **Existence of Solution(وجود جواب) :**

    Existence of solution to Inv.-Kin. problem depends on the existence of the specified goal point in the manipulator's Workspace.

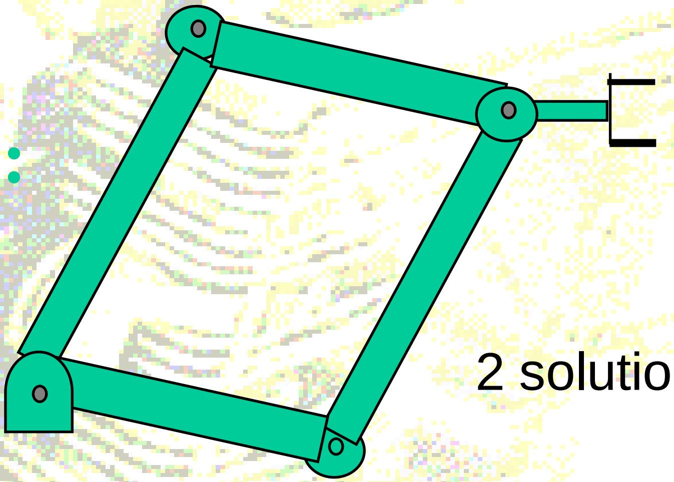
    Workspace/Work-envelope(فضای کاری) : is that volume of space which the end-effector of a robot can reach.

    Dexterous Workspace(فضای کاری ماهر) : is that volume of space which the end-effector of a robot can reach with all orientations.

Inverse Manipulator Kinematics

- **Solvability (قابل حل بودن):**
  - **Multiple Solutions(تعدد جوابها) :**

A manipulator may reach any position in the interior of its workspace with different configurations. But the system has to be able to choose one.



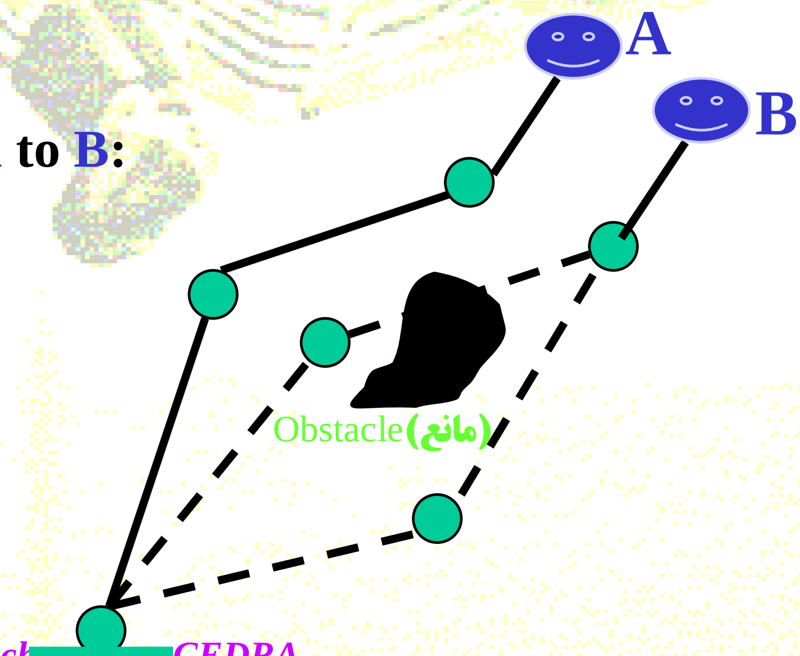
A manipulator moving from point A to B:

Two solutions exist:

- One causes a collision, and

- Other is safe.

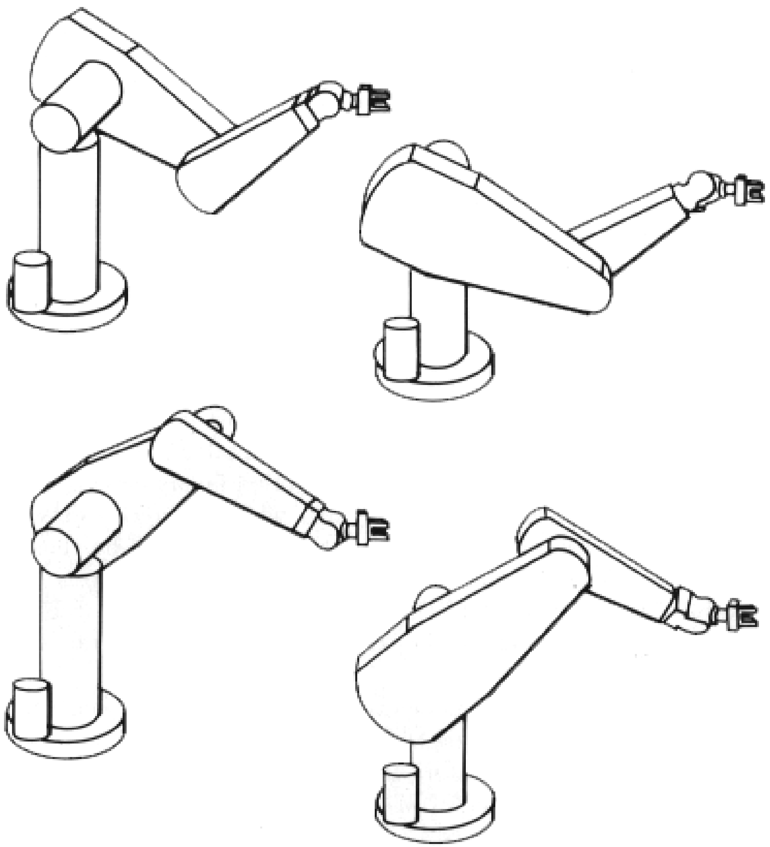
Therefore, we need to find all solutions.



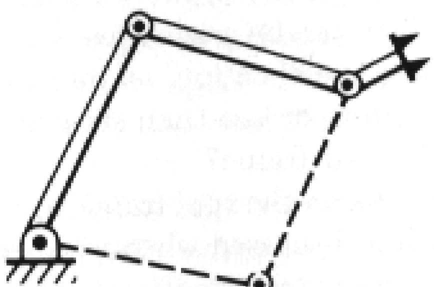
Inverse Manipulator Kinematics

- **Solvability (قابل حل بودن):**
  - **Multiple Solutions(تعدد جوابها) :**

Ex: The PUMA-560 manipulator can reach certain goals with 8-different solutions. Due to the limits imposed on joints ranges, some of these solutions may not be accessible.



{Other 4-solutions are for the wrist}



Elbow down - Elbow up

Inverse Manipulator Kinematics

- **Solvability (قابل حل بودن):**

➢ **Method of Solution(روش حل) :**

Unlike linear equations, no general algorithms exist for solving a set of *non-linear* equations.

A manipulator is considered as Solvable (قابل حل), if it is possible to calculate all its solutions. Two forms of solution strategies exist:

Closed-form-Solutions (حل بسته): Solution method is based on analytical expressions.
Numerical Solutions (حل عددی): Due to their iterative nature, they are too slow, and therefore not a useful approach in solving robot kinematics.

Inverse Manipulator Kinematics

- **Solvability (قابل حل بودن):**

➢ **Method of Solution(روش حل) :**

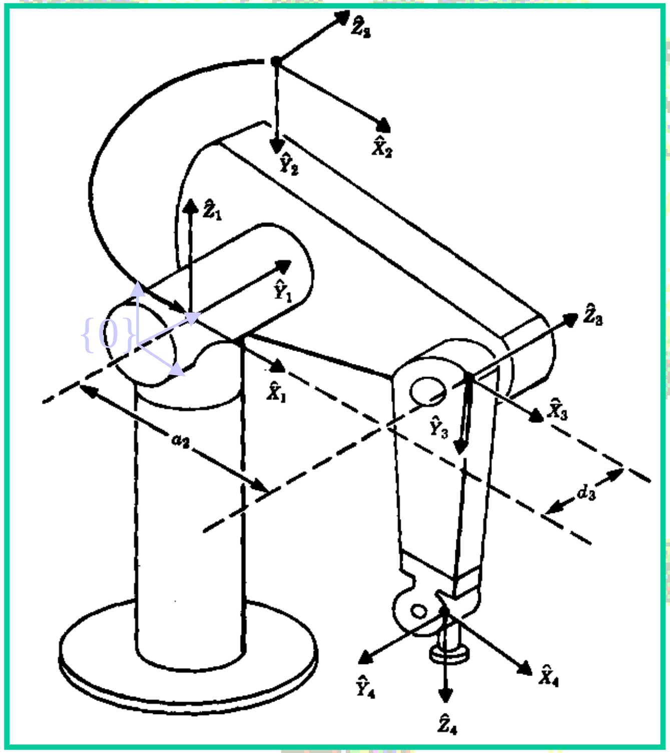
Since numerical solutions are generally very slow relative to closed form solutions, it is very important to design a manipulator such that a closed form solution exists.

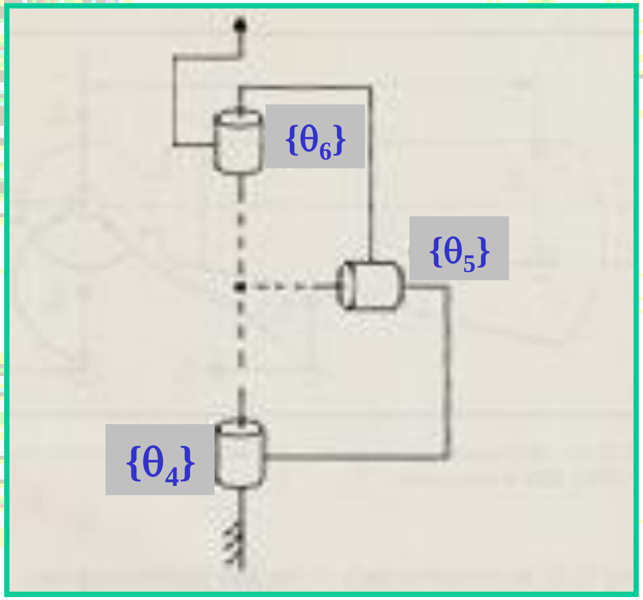
Sufficient condition for a manipulator with 6-Revolute joints to have a closed-form-solution is that 3-neighboring joints axes intersect at a point. (read section 4.6 by Pieper)

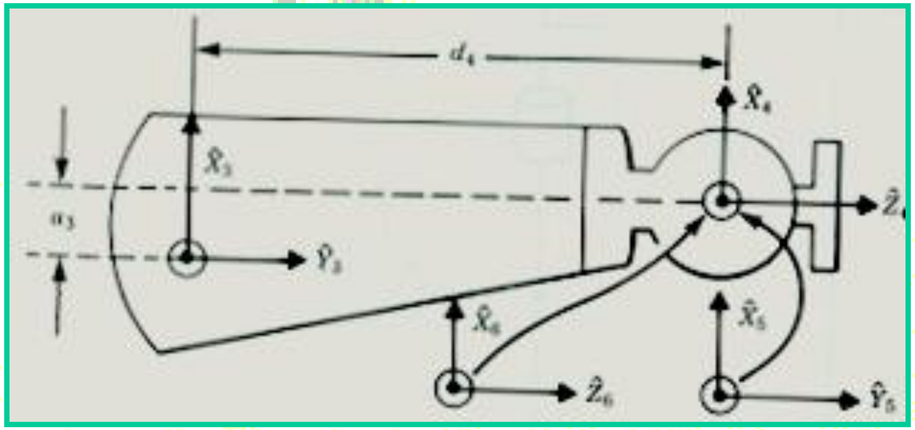
Example: In PUMA-560, axes 4, 5, and 6 all intersect at a point.

PUMA-560 Manipulator Kinematics

- **Frames Attachment (اتصال چهارچوبها):**
  - **Joint axes 4, 5, and 6 all intersect at a common point.**



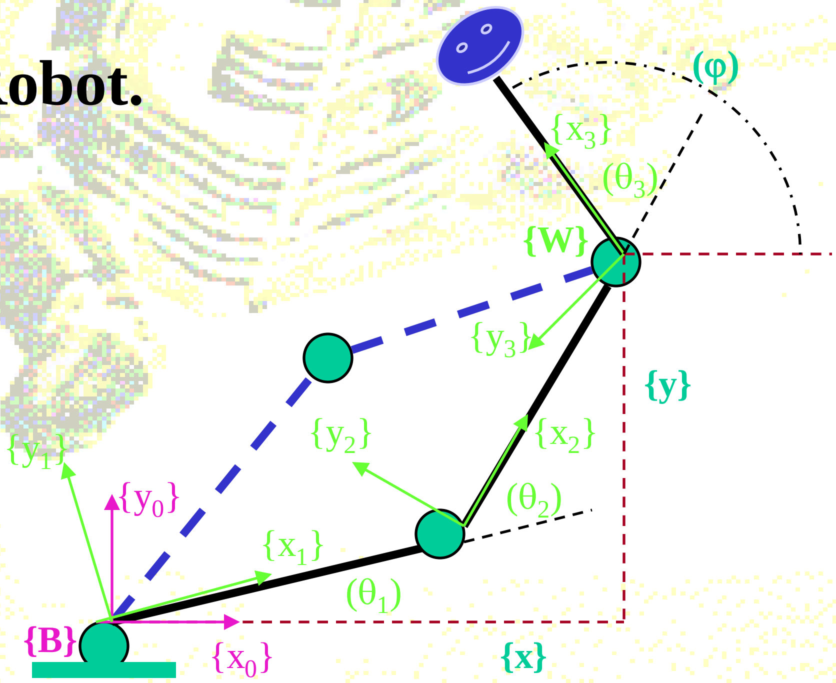




Inverse Manipulator Kinematics

- **Algebraic Method:** **No general method exists to solve kinematics equations. Let's solve a few examples.**
  - **Ex:** **A 3-DOF Revolute Planar Robot.**

Joint-i	θ_i	α_{i-1}	a_{i-1}	d_i
1	θ_1	$\alpha_0=0$	$a_0=0$	$d_1=0$
2	θ_2	$\alpha_1=0$	$a_1=L_1$	$d_2=0$
3	θ_3	$\alpha_2=0$	$a_2=L_2$	$d_3=0$



Manipulator Kinematics

- **Example:** **The 3-link planar manipulator**

$$ {}^{0}_{1}T = \begin{bmatrix} C\theta_1 & -S\theta_1 & 0 & 0 \\ S\theta_1 & C\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{1}_{2}T = \begin{bmatrix} C\theta_2 & -S\theta_2 & 0 & \ell_1 \\ S\theta_2 & C\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{2}_{3}T = \begin{bmatrix} C\theta_3 & -S\theta_3 & 0 & \ell_2 \\ S\theta_3 & C\theta_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{B}_{W}T = {}^{0}_{3}T = \begin{bmatrix} C_{123} & -S_{123} & 0 & \ell_1 C_1 + \ell_2 C_{12} \\ S_{123} & C_{123} & 0 & \ell_1 S_1 + \ell_2 S_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{(a)} $$

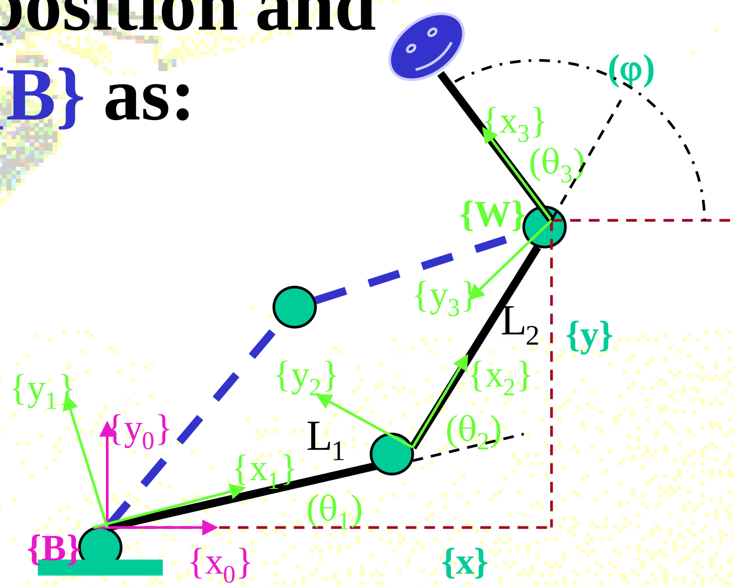
Since this is a planar robot, assume that the goal point is a specification of the {Wrist} relative to the {Base}.

Inverse Manipulator Kinematics

- **Therefore, we can use 3-numbers x, y, and φ to specify the goal point such that:**
  - **x, y: define the origin of frame {W}, and**
  - **φ: defines the orientation of {W} (3rd-link) relative to the +x axis of the {B} frame.**
  - **Therefore, one can define the position and orientation of {W} relative to {B} as:**

$$
{}^{B}_{W}T = {}^{0}_{3}T = \begin{bmatrix} C\varphi & -S\varphi & 0 & x \\ S\varphi & C\varphi & 0 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

(b)



Inverse Manipulator Kinematics

- **Let use now equate relations (a) and (b) as follows:**

$$ {}^{B}_{W}T = {}^{0}_{3}T = \begin{bmatrix} C_{123} & -S_{123} & 0 & \ell_1 C_1 + \ell_2 C_{12} \\ S_{123} & C_{123} & 0 & \ell_1 S_1 + \ell_2 S_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} C\varphi & -S\varphi & 0 & x \\ S\varphi & C\varphi & 0 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$(1) \quad C\varphi = C_{123}$$

$$(2) \quad S\varphi = S_{123}$$

$$(3) \quad x = \ell_1 C_1 + \ell_2 C_{12}$$

$$(4) \quad y = \ell_1 S_1 + \ell_2 S_{12}$$

$$Square - and - Add - (3) \,\&\, (4):$$

$$x^2 + y^2 = \ell_1^2 + \ell_2^2 + 2\ell_1 \ell_2 (C_1 C_{12} + S_1 S_{12})$$

$$Since: (C_1 C_{12} + S_1 S_{12}) = C_1 (C_1 C_2 - S_1 S_2) + S_1 (S_1 C_2 + C_1 S_2) = C_2$$

$$x^2 + y^2 = \ell_1^2 + \ell_2^2 + 2\ell_1 \ell_2 C_2$$

Inverse Manipulator Kinematics

- **Using Atan2 function insures finding all solutions.**

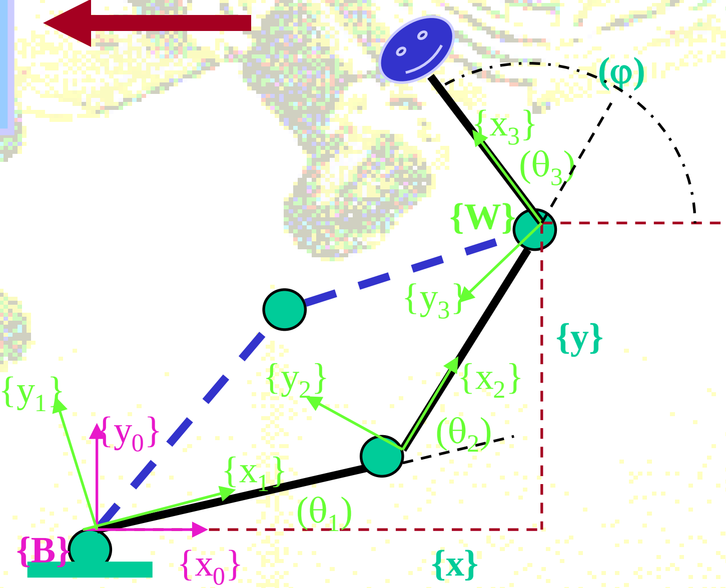
$$C_2 = \frac{x^2 + y^2 - \ell_1^2 - \ell_2^2}{2\ell_1\ell_2}$$

$$S_2 = \pm\sqrt{1 - C_2^2}$$

$$\theta_2 = A\tan 2\left(\frac{S_2}{C_2}\right)$$

(This term should be between –1 and 1. If it is not, that means we have an unreachable point.)

(+ and – means Multiple Solutions for θ_2: Elbow-Up and Elbow-Down configurations.)



Inverse Manipulator Kinematics

- To find θ_1 use equations (3) and (4) as follows:

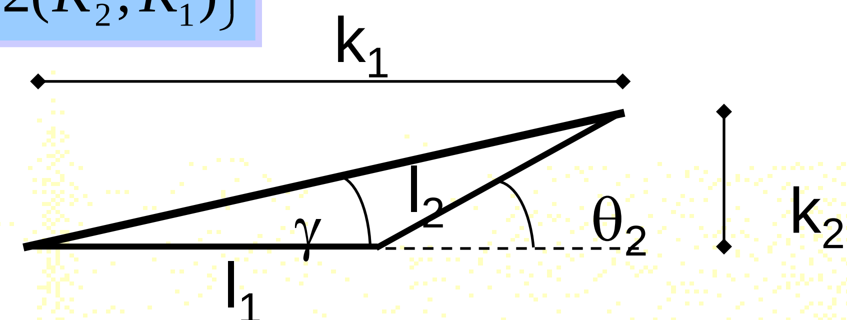
$$(3) \qquad x = \ell_1 C_1 + \ell_2 C_{12} = \ell_1 C_1 + \ell_2 C_1 C_2 - \ell_2 S_1 S_2 =$$

$$x = (\ell_1 + \ell_2 C_2) C_1 - (\ell_2 S_2) S_1 = K_1 C_1 - K_2 S_1$$

$$(4) \qquad y = \ell_1 S_1 + \ell_2 S_{12} = K_1 S_1 + K_2 C_1$$

- Let us now change variables to solve these equations:

$$Let: \begin{Bmatrix} K_1 = r\cos\gamma \\ K_2 = r\sin\gamma \end{Bmatrix} \Leftrightarrow \begin{Bmatrix} r = +\sqrt{K_1^2 + K_2^2} \\ \gamma = A\tan 2(K_2, K_1) \end{Bmatrix}$$



- Now relations for **x** and **y** can be expressed as:

Inverse Manipulator Kinematics

- **Now relations for x and y can be expressed as:**

$$x = r\cos\gamma\cos\theta_1 - r\sin\gamma\sin\theta_1 \Rightarrow \frac{x}{r} = \cos(\gamma+\theta_1)$$

$$y = r\cos\gamma\sin\theta_1 + r\sin\gamma\cos\theta_1 \Rightarrow \frac{y}{r} = \sin(\gamma+\theta_1)$$

- **Therefore:**

$$\gamma+\theta_1 = A\tan 2(\frac{y}{r},\frac{x}{r}) = A\tan 2(y,x) \Rightarrow$$

$$\theta_1 = A\tan 2(y,x) - A\tan 2(K_2,K_1)$$

- **One solution for θ_1, and that depends on the sign chosen for θ_2.**
- **From equations (1) and (2) , we can now define θ_3.**

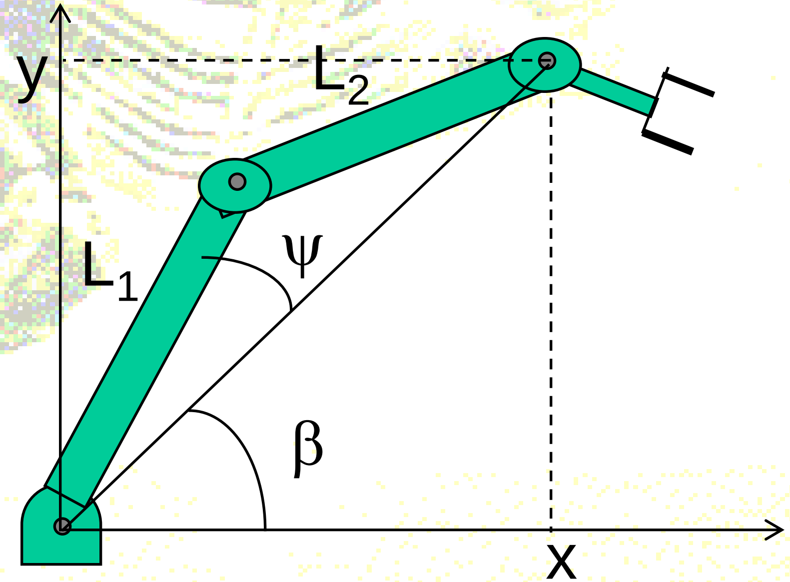
$$\begin{Bmatrix} C\varphi = C_{123} \\ S\varphi = S_{123} \end{Bmatrix} \Rightarrow \theta_{123} = \theta_1+\theta_2+\theta_3 = A\tan 2(\frac{S\varphi}{C\varphi}) = \varphi \Rightarrow$$

$$\theta_3 = \varphi - \theta_1 - \theta_2$$

Inverse Manipulator Kinematics

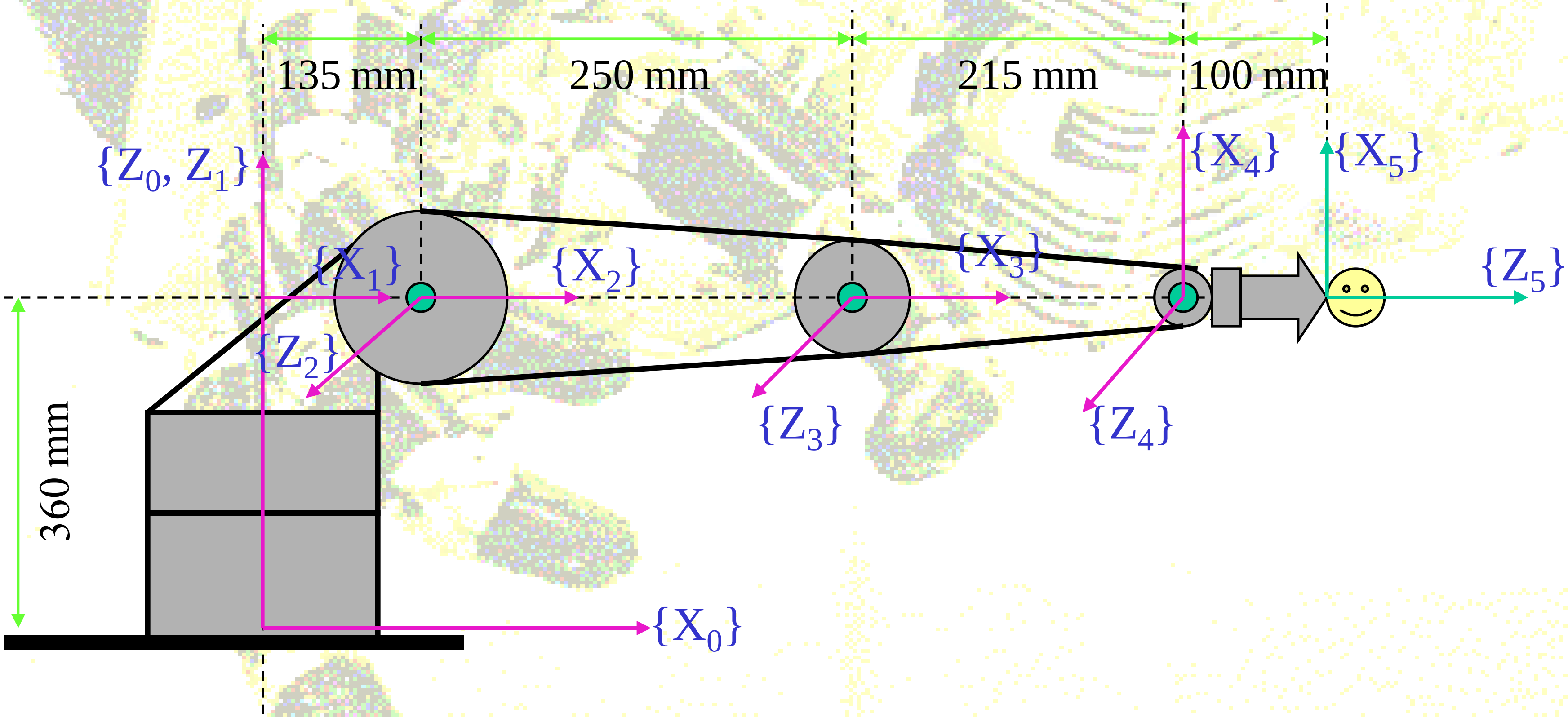
- **Geometric Method:** **First decompose the spatial geometry of the arm into several plane geometry problems. Then, solve for the joint angles using tools of plane geometry (i.e by applying the "law of cosines"). (see book for an example)**

Joint-i	θ_i	α_{i-1}	a_{i-1}	d_i
1	θ_1	$\alpha_0=0$	$a_0=0$	$d_1=0$
2	θ_2	$\alpha_1=0$	$a_1=L_1$	$d_2=0$
3	θ_3	$\alpha_2=0$	$a_2=L_2$	$d_3=0$



Manipulator Kinematics

- **Example:** **The Yasukawa/Motoman MK3 Robot.**
  - **A 5-DOF "5R" Revolute Robot**



Manipulator Kinematics

- The Yasukawa/Motoman MK3 Table of Link-Joint Parameters:

Joint-i	${}^{i-1}_{i}T$	θ_i	α_{i-1}	a_{i-1}	d_i
1	${}^{0}_{1}T$	θ_1	$\alpha_0 = 0$	$a_0 = 0$	$d_1 = 360$
2	${}^{1}_{2}T$	θ_2	$\alpha_1 = 90$	$a_1 = 135$	$d_2 = 0$
3	${}^{2}_{3}T$	θ_3	$\alpha_2 = 0$	$a_2 = 250$	$d_3 = 0$
4	${}^{3}_{4}T$	θ_4	$\alpha_3 = 0$	$a_3 = 215$	$d_4 = 0$
5	${}^{4}_{5}T$	θ_5	$\alpha_4 = 90$	$a_4 = 0$	$d_5 = 100$

Yasukawa/Motoman MK3 Manipulator Kinematics

- Now compute each of the link *transformations*:

$$ {}^0_1T = \begin{bmatrix} C_1 & -S_1 & 0 & 0 \\ S_1 & C_1 & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^1_2T = \begin{bmatrix} C_2 & -S_2 & 0 & a_1 \\ 0 & 0 & -1 & 0 \\ S_2 & C_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^2_3T = \begin{bmatrix} C_3 & -S_3 & 0 & a_2 \\ S_3 & C_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^3_4T = \begin{bmatrix} C_4 & -S_4 & 0 & a_3 \\ S_4 & C_4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^4_5T = \begin{bmatrix} C_5 & -S_5 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ S_5 & C_5 & 0 & -d_5 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

YM MK3 Manipulator Kinematics

- **Let us now form the** ${}^{0}_{5}T$ ***transformation matrix*:**

$$ {}^{0}_{5}T = {}^{0}_{1}T\,{}^{1}_{2}T\,{}^{2}_{3}T\,{}^{3}_{4}T\,{}^{4}_{5}T = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ r_{11} = C_1 C_{234} C_5 + S_1 S_5 \qquad r_{12} = -C_1 C_{234} S_5 + S_1 C_5 \qquad r_{13} = C_1 S_{234} $$

$$ r_{21} = S_1 C_{234} C_5 - C_1 S_5 \qquad r_{22} = -S_1 C_{234} S_5 - C_1 C_5 \qquad r_{23} = S_1 S_{234} $$

$$ r_{31} = C_5 S_{234} \qquad r_{32} = -S_5 S_{234} \qquad r_{33} = -C_{234} $$

$$ p_x = C_1(a_1 + a_2 C_2 + a_3 C_{23} + d_5 S_{234}) $$

$$ p_y = S_1(a_1 + a_2 C_2 + a_3 C_{23} + d_5 S_{234}) $$

$$ p_z = (d_1 + a_2 S_2 + a_3 S_{23} - d_5 C_{234}) $$

YM MK3 Manipulator Inverse-Kinematics

- We wish to solve the inverse-kinematics problem yielding $\theta_1 \ldots \theta_5$ as a function of $r_{11} \ldots r_{33}$, p_x, p_y, p_z.
- Lets start with θ_1, since no "yaw" motion is present:

$$\theta_1 = A\tan 2\left(p_y / p_x\right) \qquad Since \quad \begin{cases} p_x = C_1(a_1 + a_2C_2 + a_3C_{23} + d_5S_{234}) \\ p_y = S_1(a_1 + a_2C_2 + a_3C_{23} + d_5S_{234}) \end{cases}$$

- Note that we cannot have Atan2 (0/0) !!! If: $p_x = p_y = 0$, then we have a special case.
- We now need $(\theta_2 + \theta_3 + \theta_4)$ to find the wrist center. Note that:

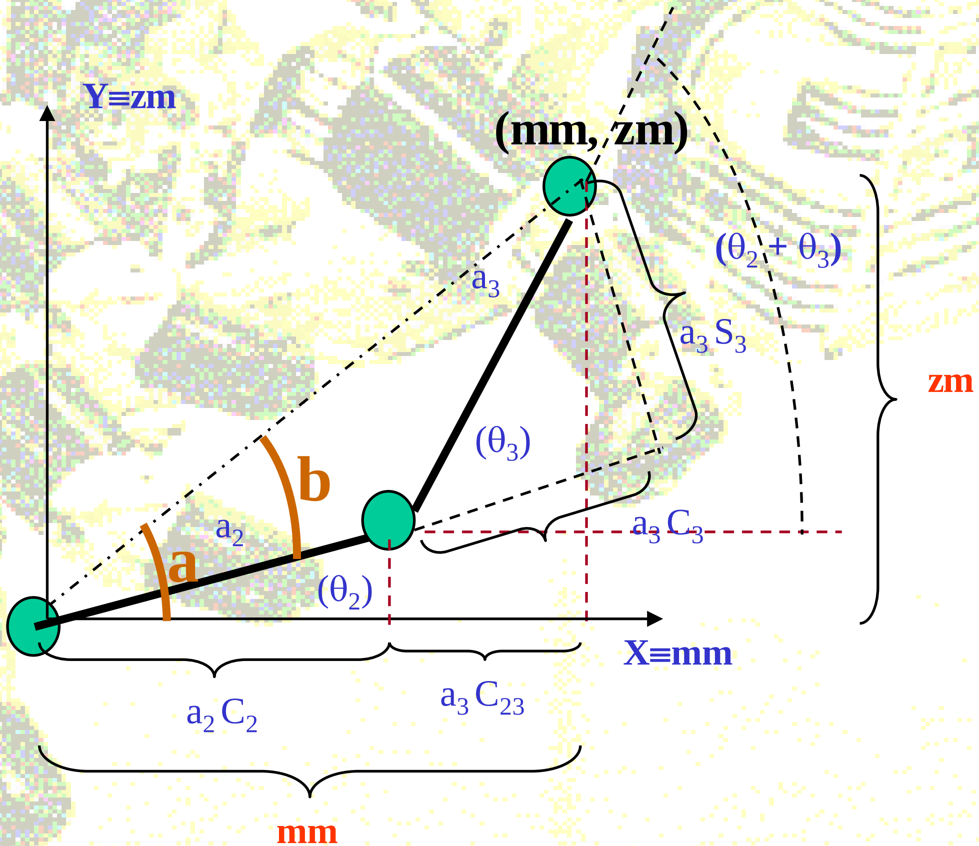
$$r_{13} = C_1S_{234}, \qquad r_{23} = -S_1S_{234}, \qquad r_{33} = -C_{234}$$

- Therefore, we can write:

$$C_1r_{13} - S_1r_{23} = C_1^2S_{234} + S_1^2S_{234} = S_{234} \Rightarrow$$
$$\theta_2 + \theta_3 + \theta_4 = A\tan 2(C_1r_{13} - S_1r_{23}, -r_{33})$$

YM MK3 Manipulator Inverse-Kinematics

- Let's now solve for θ_2 and θ_3 as follows (by reconsidering our old planar arm problem):



YM MK3 Manipulator Inverse-Kinematics

$$Let: \quad m = C_1 p_x + S_1 p_y = C_1^2(\ldots) + S_1^2(\ldots) = (a_1 + a_2 C_2 + a_3 C_{23} + d_5 S_{234})$$

$$Let: \quad X = mm = m - a_1 - d_5 S_{234} = a_2 C_2 + a_3 C_{23}, \quad S_{234} = known$$

$$Let: \quad Y = zm = p_z - d_1 + d_5 C_{234} = a_2 S_2 + a_3 S_{23}, \quad C_{234} = known$$

$Then:$

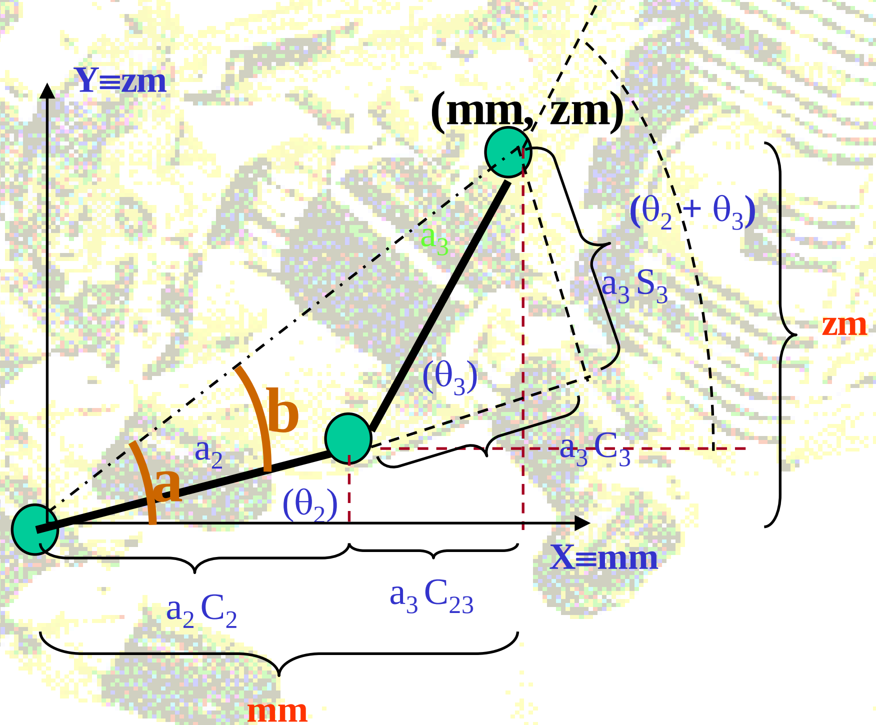
$$\begin{aligned} mm^2 + zm^2 &= a_2^2 C_2^2 + a_3^2 C_{23}^2 + 2a_2 a_3 C_2 C_{23} + \\ &+ a_2^2 S_2^2 + a_3^2 S_{23}^2 + 2a_2 a_3 S_2 S_{23} = \\ &= a_2^2 + a_3^2 + 2a_2 a_3 (C_2 C_{23} + S_2 S_{23}) = a_2^2 + a_3^2 + 2a_2 a_3 (C_3) \end{aligned}$$

$$Cos\theta_3 = \frac{mm^2 + zm^2 - a_2^2 - a_3^2}{2a_2 a_3}, and \quad Sin\theta_3 = \pm\sqrt{1 - C_3^2}$$

$$\theta_3 = A\tan 2(\frac{S_3}{C_3})$$

YM MK3 Manipulator Inverse-Kinematics

- **Considering the following figure again, we have:**



$$\left\{ \begin{array}{c} a = A\tan 2(zm, mm) \\ b = A\tan 2(a_3 S_3, a_2 + a_3 C_3) \end{array} \right\} \Rightarrow \theta_2 = a - b$$

YM MK3 Manipulator Inverse-Kinematics

- Solving for θ_4 we have:

$$\theta_4 = (\theta_2 + \theta_3 + \theta_4) - \theta_2 - \theta_3$$

- Finally, to solve for θ_5, given $\theta_1 \ldots \theta_4$, note that ${}^0_4T$ is now known. Therefore, we can write the following equation:

$${}^0_5T = {}^0_4T\,{}^4_5T \Rightarrow \qquad {}^4_5T = {}^0_4T^{-1}\,{}^0_5T \qquad \Rightarrow$$

$$\begin{bmatrix} C_5 & -S_5 & 0 & 0 \\ 0 & 0 & -1 & -d_5 \\ S_5 & C_5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} C_1C_{234} & S_1C_{234} & S_{234} & \ldots \\ -C_1S_{234} & -S_1S_{234} & C_{234} & \ldots \\ S_1 & -C_1 & 0 & \ldots \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Hence: \begin{Bmatrix} S_5 = S_1 r_{11} - C_1 r_{21} \\ C_5 = S_1 r_{12} - C_1 r_{22} \end{Bmatrix} \Rightarrow \theta_5 = A\tan 2(S_5, C_5)$$

Chapter 4 Exercises:

- 4.1, 4.2, 4.3, 4.8, 4.9
- 4.1 Programming Exercise
- 4.1 MathLab Exercise
- Programming of the PUMA 560 Inverse Kinematics