SESSION # 13: MANIPULATOR INVERSE KINEMATICS

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INTRODUCTION TO ROBOTICS
(Kinematics, Dynamics, and Design)
Inverse Manipulator Kinematics

- **Forward Kinematics**: Describe the position and orientation of the manipulator’s end-effector as a function of joint variables relative to a base frame.

- **Inverse Kinematics**: Given the desired position and orientation of the end-effector relative to the base, compute the set of joint variables which will achieve this desired result.

A 3-DOF Manipulator Arm
Inverse Manipulator Kinematics

- Solvability (قابل حل بودن): Solving kinematics equations in robotics is a Non-Linear Problem.

Given: \( ^0T_n \), Find; \( \{\theta_1, \theta_2, ..., \theta_n\} \), is a non-linear problem.

**Ex:** PUMA-560 Robot. Given; \( ^0T_6 \), Find; \( \{\theta_1, \theta_2, ..., \theta_6\} \), (see Equation 3.14)

For a 6-DOF manipulator, we have:

- 12-Equations, and 6-Unknowns?
- From 9-Equations of the Rotation Matrix, only 3-Equations are independent.
- Therefore, we have 6-independent non-linear equations and 6-unknowns.
Inverse Manipulator Kinematics

• Solvability (قابل حل بودن):

We have 6-independent non-linear equations and 6-unknowns. Therefore, we should investigate the followings:

- Existence of Solution (وجود جواب).
- Multiple Solutions (تعدد جوابها).
- Method of Solution (روش حل).
Inverse Manipulator Kinematics

• Solvability (قابل حل بودن):

Existence of Solution (وجود جواب):
Existence of solution to Inv.-Kin. problem depends on the existence of the specified goal point in the manipulator’s Workspace.

Workspace/Work-envelope (فضای کاری): is that volume of space which the end-effector of a robot can reach.
Dexterous Workspace (فضای کاری ماهر): is that volume of space which the end-effector of a robot can reach with all orientations.
Inverse Manipulator Kinematics

- Solvability (قابل حل بودن):

Multiple Solutions (تعدد جوابها):

A manipulator may reach any position in the interior of its workspace with different configurations. But the system has to be able to choose one.

A manipulator moving from point A to B:

Two solutions exist:
- One causes a collision, and
- Other is safe.

Therefore, we need to find all solutions.
Inverse Manipulator Kinematics

- Solvability (قابل حل بودن):
  
  **Multiple Solutions** (تعداد جوابها):

  **Ex:** The PUMA-560 manipulator can reach certain goals with 8-different solutions. Due to the limits imposed on joints ranges, some of these solutions may not be accessible.

  {Other 4-solutions are for the wrist}

Elbow down - Elbow up
Inverse Manipulator Kinematics

- Solvability (قابل حل بودن):

  ➢ Method of Solution (روش حل):

Unlike linear equations, no general algorithms exist for solving a set of non-linear equations.

A manipulator is considered as Solvable (قابل حل), if it is possible to calculate all its solutions. Two forms of solution strategies exist:

Closed-form-Solutions (حل بسته): Solution method is based on analytical expressions.

Numerical Solutions (حل عددي): Due to their iterative nature, they are too slow, and therefore not a useful approach in solving robot kinematics.
Inverse Manipulator Kinematics

• Solvability (قابل حل بودن):

  ➢ Method of Solution (روش حل):

Since numerical solutions are generally very slow relative to closed form solutions, it is very important to design a manipulator such that a closed form solution exists.

Sufficient condition for a manipulator with 6-Revolute joints to have a closed-form-solution is that 3-neighboring joints axes intersect at a point. (read section 4.6 by Pieper)

Example: In PUMA-560, axes 4, 5, and 6 all intersect at a point.
PUMA-560 Manipulator Kinematics

- Frames Attachment (اتصال چهار چوبها):
  - Joint axes 4, 5, and 6 all intersect at a common point.
Inverse Manipulator Kinematics

- **Algebraic Method:** No general method exists to solve kinematics equations. Let’s solve a few examples.

  - **Ex:** A 3-DOF Revolute Planar Robot.

<table>
<thead>
<tr>
<th>Joint-i</th>
<th>$\theta_i$</th>
<th>$\alpha_{i-1}$</th>
<th>$a_{i-1}$</th>
<th>$d_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\theta_1$</td>
<td>$\alpha_0=0$</td>
<td>$a_0=0$</td>
<td>$d_1=0$</td>
</tr>
<tr>
<td>2</td>
<td>$\theta_2$</td>
<td>$\alpha_1=0$</td>
<td>$a_1=L_1$</td>
<td>$d_2=0$</td>
</tr>
<tr>
<td>3</td>
<td>$\theta_3$</td>
<td>$\alpha_2=0$</td>
<td>$a_2=L_2$</td>
<td>$d_3=0$</td>
</tr>
</tbody>
</table>
Manipulator Kinematics

Example: The 3-link planar manipulator

\[
0^T = \begin{bmatrix}
C\theta_1 & -S\theta_1 & 0 & 0 \\
S\theta_1 & C\theta_1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
1^T = \begin{bmatrix}
C\theta_2 & -S\theta_2 & 0 & \ell_1 \\
S\theta_2 & C\theta_2 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
2^T = \begin{bmatrix}
C\theta_3 & -S\theta_3 & 0 & \ell_2 \\
S\theta_3 & C\theta_3 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
B^T W = 0^T T^0_3 = \begin{bmatrix}
C_{123} & -S_{123} & 0 & \ell_1 C_1 + \ell_2 C_{12} \\
S_{123} & C_{123} & 0 & \ell_1 S_1 + \ell_2 S_{12} \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

(a)

Since this is a planar robot, assume that the goal point is a specification of the \{Wrist\} relative to the \{Base\}. 
Inverse Manipulator Kinematics

- Therefore, we can use 3-numbers \( x, y, \) and \( \varphi \) to specify the goal point such that:
  - \( x, y \): define the origin of frame \( \{W\} \), and
  - \( \varphi \): defines the orientation of \( \{W\} \) (3rd-link) relative to the +x axis of the \( \{B\} \) frame.
  - Therefore, one can define the position and orientation of \( \{W\} \) relative to \( \{B\} \) as:

\[
B^W_T = ^0_T = \begin{bmatrix}
C\varphi & -S\varphi & 0 & x \\
S\varphi & C\varphi & 0 & y \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
Inverse Manipulator Kinematics

Let use now equate relations (a) and (b) as follows:

\[
\begin{aligned}
B \mathbf{T}^0_w &= \begin{bmatrix}
C_{123} & -S_{123} & 0 & \ell_1 C_1 + \ell_2 C_{12} \\
S_{123} & C_{123} & 0 & \ell_1 S_1 + \ell_2 S_{12} \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix} = \begin{bmatrix}
C \phi & -S \phi & 0 & x \\
S \phi & C \phi & 0 & y \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\end{aligned}
\]

(1) \( C \phi = C_{123} \)
(2) \( S \phi = S_{123} \)
(3) \( x = \ell_1 C_1 + \ell_2 C_{12} \)
(4) \( y = \ell_1 S_1 + \ell_2 S_{12} \)

Square – and – Add – (3) & (4):

\[
x^2 + y^2 = \ell_1^2 + \ell_2^2 + 2 \ell_1 \ell_2 (C_1 C_{12} + S_1 S_{12})
\]

Since: \((C_1 C_{12} + S_1 S_{12}) = C_1 (C_1 C_2 - S_1 S_2) + S_1 (S_1 C_2 + C_1 S_2) = C_2 \)

\[
x^2 + y^2 = \ell_1^2 + \ell_2^2 + 2 \ell_1 \ell_2 C_2
\]
Inverse Manipulator Kinematics

Using Atan2 function insures finding all solutions.

\[ C_2 = \frac{x^2 + y^2 - \ell_1^2 - \ell_2^2}{2\ell_1\ell_2} \]

\[ S_2 = \pm \sqrt{1 - C_2^2} \]

\[ \theta_2 = A \tan 2\left(\frac{S_2}{C_2}\right) \]

(This term should be between \(-1\) and 1. If it is not, that means we have an unreachable point.)

(\(+\) and \(-\) means Multiple Solutions for \(\theta_2\): Elbow-Up and Elbow-Down configurations.)
Inverse Manipulator Kinematics

To find $\theta_1$ use equations (3) and (4) as follows:

\[ x = \ell_1 C_1 + \ell_2 C_{12} = \ell_1 C_1 + \ell_2 C_1 C_2 - \ell_2 S_1 S_2 = x = (\ell_1 + \ell_2 C_2)C_1 - (\ell_2 S_2)S_1 = K_1 C_1 - K_2 S_1 \]

\[ y = \ell_1 S_1 + \ell_2 S_{12} = K_1 S_1 + K_2 C_1 \]

Let us now change variables to solve these equations:

\[ \begin{align*}
K_1 &= r \cos \gamma \\
K_2 &= r \sin \gamma \\
r &= +\sqrt{K_1^2 + K_2^2} \\
\gamma &= A \tan 2(K_2, K_1)
\end{align*} \]

Now relations for $x$ and $y$ can be expressed as:
Inverse Manipulator Kinematics

Now relations for \( x \) and \( y \) can be expressed as:

\[
x = r \cos \gamma \cos \theta_1 - r \sin \gamma \sin \theta_1 \Rightarrow \frac{x}{r} = \cos(\gamma + \theta_1)
\]

\[
y = r \cos \gamma \sin \theta_1 + r \sin \gamma \cos \theta_1 \Rightarrow \frac{y}{r} = \sin(\gamma + \theta_1)
\]

Therefore:

\[
\gamma + \theta_1 = A \tan 2\left(\frac{y}{r}, \frac{x}{r}\right) = A \tan 2(y, x) \Rightarrow \theta_1 = A \tan 2(y, x) - A \tan 2(K_2, K_1)
\]

One solution for \( \theta_1 \), and that depends on the sign chosen for \( \theta_2 \).

From equations (1) and (2), we can now define \( \theta_3 \).

\[
\begin{align*}
C \varphi &= C_{123} \\
S \varphi &= S_{123}
\end{align*}

\Rightarrow \theta_{123} = \theta_1 + \theta_2 + \theta_3 = A \tan 2\left(\frac{S \varphi}{C \varphi}\right) = \varphi \Rightarrow \theta_3 = \varphi - \theta_1 - \theta_2
\]
Inverse Manipulator Kinematics

- **Geometric Method:** First decompose the spatial geometry of the arm into several plane geometry problems. Then, solve for the joint angles using tools of plane geometry (i.e. by applying the “law of cosines”). (see book for an example)

<table>
<thead>
<tr>
<th>Joint-i</th>
<th>( \theta_i )</th>
<th>( \alpha_{i-1} )</th>
<th>( a_{i-1} )</th>
<th>( d_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \theta_1 )</td>
<td>( \alpha_0=0 )</td>
<td>( a_0=0 )</td>
<td>( d_1=0 )</td>
</tr>
<tr>
<td>2</td>
<td>( \theta_2 )</td>
<td>( \alpha_1=0 )</td>
<td>( a_1=L_1 )</td>
<td>( d_2=0 )</td>
</tr>
<tr>
<td>3</td>
<td>( \theta_3 )</td>
<td>( \alpha_2=0 )</td>
<td>( a_2=L_2 )</td>
<td>( d_3=0 )</td>
</tr>
</tbody>
</table>
Manipulator Kinematics

- **Example:** The Yasukawa/Motoman MK3 Robot.
  - A 5-DOF “5R” Revolute Robot

![Diagram of a 5-DOF Revolute Robot]

- Dimensions:
  - $L_1 = 135$ mm
  - $L_2 = 250$ mm
  - $L_3 = 215$ mm
  - $L_4 = 100$ mm

- Coordinate frames:
  - $\{X_0\}$
  - $\{Z_0, Z_1\}$
  - $\{X_1\}$
  - $\{X_2\}$
  - $\{X_3\}$
  - $\{X_4\}$
  - $\{Z_4\}$
  - $\{X_5\}$
  - $\{Z_5\}$

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## Manipulator Kinematics

- **The Yasukawa/Motoman MK3 Table of Link-Joint Parameters:**

<table>
<thead>
<tr>
<th>Joint-i</th>
<th>$i-1T$</th>
<th>$\theta_i$</th>
<th>$\alpha_{i-1}$</th>
<th>$a_{i-1}$</th>
<th>$d_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$0^1T$</td>
<td>$\theta_1$</td>
<td>$\alpha_0=0$</td>
<td>$a_0=0$</td>
<td>$d_1=360$</td>
</tr>
<tr>
<td>2</td>
<td>$1^2T$</td>
<td>$\theta_2$</td>
<td>$\alpha_1=90$</td>
<td>$a_1=135$</td>
<td>$d_2=0$</td>
</tr>
<tr>
<td>3</td>
<td>$2^3T$</td>
<td>$\theta_3$</td>
<td>$\alpha_2=0$</td>
<td>$a_2=250$</td>
<td>$d_3=0$</td>
</tr>
<tr>
<td>4</td>
<td>$3^4T$</td>
<td>$\theta_4$</td>
<td>$\alpha_3=0$</td>
<td>$a_3=215$</td>
<td>$d_4=0$</td>
</tr>
<tr>
<td>5</td>
<td>$4^5T$</td>
<td>$\theta_5$</td>
<td>$\alpha_4=90$</td>
<td>$a_4=0$</td>
<td>$d_5=100$</td>
</tr>
</tbody>
</table>
Yasukawa/Motoman MK3 Manipulator Kinematics

• Now compute each of the link transformations:

$$^{0}_{1}T = \begin{bmatrix} C_1 & -S_1 & 0 & 0 \\ S_1 & C_1 & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^{1}_{2}T = \begin{bmatrix} C_2 & -S_2 & 0 & a_1 \\ S_2 & C_2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^{2}_{3}T = \begin{bmatrix} C_3 & -S_3 & 0 & a_2 \\ S_3 & C_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^{3}_{4}T = \begin{bmatrix} C_4 & -S_4 & 0 & a_3 \\ S_4 & C_4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^{4}_{5}T = \begin{bmatrix} C_5 & -S_5 & 0 & 0 \\ S_5 & C_5 & 0 & -d_5 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
YM MK3 Manipulator Kinematics

- Let us now form the transformation matrix:

\[
0^T_5 = 0^T_1 T_2^T T_3^T T_4^T T_5^T = \begin{bmatrix}
  r_{11} & r_{12} & r_{13} & p_x \\
  r_{21} & r_{22} & r_{23} & p_y \\
  r_{31} & r_{32} & r_{33} & p_z \\
  0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
\begin{align*}
  r_{11} &= C_1 C_{234} C_5 + S_1 S_5 \\
  r_{12} &= -C_1 C_{234} S_5 + S_1 C_5 \\
  r_{13} &= C_1 S_{234} \\
  r_{21} &= S_1 C_{234} C_5 - C_1 S_5 \\
  r_{22} &= -S_1 C_{234} S_5 - C_1 C_5 \\
  r_{23} &= S_1 S_{234} \\
  r_{31} &= C_5 S_{234} \\
  r_{32} &= -S_5 S_{234} \\
  r_{33} &= -C_{234} \\
  p_x &= C_1 (a_1 + a_2 C_2 + a_3 C_{23} + d_5 S_{234}) \\
  p_y &= S_1 (a_1 + a_2 C_2 + a_3 C_{23} + d_5 S_{234}) \\
  p_z &= (d_1 + a_2 S_2 + a_3 S_{23} - d_5 C_{234})
\end{align*}
\]
YM MK3 Manipulator
Inverse-Kinematics

- We wish to solve the inverse-kinematics problem yielding $\theta_1...\theta_5$ as a function of $r_{11}...r_{33}$, $p_x$, $p_y$, $p_z$.
- Let's start with $\theta_1$, since no “yaw” motion is present:

$$\theta_1 = A \tan 2\left(\frac{p_y}{p_x}\right) \quad \text{Since} \quad \begin{cases} p_x = C_1(a_1 + a_2 C_2 + a_3 C_{23} + d_5 S_{234}) \\ p_y = S_1(a_1 + a_2 C_2 + a_3 C_{23} + d_5 S_{234}) \end{cases}$$

- Note that we cannot have $\text{Atan2} (0/0)$ !!! If: $p_x = p_y = 0$, then we have a special case.
- We now need $(\theta_2 + \theta_3 + \theta_4)$ to find the wrist center. Note that:

$$r_{13} = C_1 S_{234}, \quad r_{23} = -S_1 S_{234}, \quad r_{33} = -C_{234}$$

- Therefore, we can write:

$$C_1 r_{13} - S_1 r_{23} = C_1^2 S_{234} + S_1^2 S_{234} = S_{234} \Rightarrow \theta_2 + \theta_3 + \theta_4 = A \tan 2(C_1 r_{13} - S_1 r_{23}, -r_{33})$$
YM MK3 Manipulator
Inverse-Kinematics

Let’s now solve for $\theta_2$ and $\theta_3$ as follows (by reconsidering our old planar arm problem):

Let's solve for $\theta_2$ and $\theta_3$ as follows (by reconsidering our old planar arm problem):
YM MK3 Manipulator
Inverse-Kinematics

Let: \( m = C_1 p_x + S_1 p_y = C_1 (...) + S_1 (...) = (a_1 + a_2 C_2 + a_3 C_{23} + d_5 S_{234}) \)
Let: \( X = mm = m - a_1 - d_5 S_{234} = a_2 C_2 + a_3 C_{23}, \quad S_{234} = \text{known} \)
Let: \( Y = zm = p_z - d_1 + d_5 C_{234} = a_2 S_2 + a_3 S_{23}, \quad C_{234} = \text{known} \)

Then:

\[
mm^2 + zm^2 = a_2^2 C_2^2 + a_3^2 C_{23}^2 + 2a_2 a_3 C_2 C_{23} + \\
+ a_2^2 S_2^2 + a_3^2 S_{23}^2 + 2a_2 a_3 S_2 S_{23} = \\
= a_2^2 + a_3^2 + 2a_2 a_3 (C_2 C_{23} + S_2 S_{23}) = a_2^2 + a_3^2 + 2a_2 a_3 (C_3)
\]

\[
\cos \theta_3 = \frac{mm^2 + zm^2 - a_2^2 - a_3^2}{2a_2 a_3}, \quad \text{and} \quad \sin \theta_3 = \pm \sqrt{1 - C_3^2}
\]

\[
\theta_3 = A \tan 2 \left( \frac{S_3}{C_3} \right)
\]
YM MK3 Manipulator
Inverse-Kinematics

Considering the following figure again, we have:

\[
\begin{align*}
\theta_2 &= a - b \\
A &= \tan 2(zm, mm) \\
b &= A \tan 2(a_3S_3, a_2 + a_3C_3)
\end{align*}
\]

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YM MK3 Manipulator
Inverse-Kinematics

Solving for $\theta_4$ we have:

$$\theta_4 = (\theta_2 + \theta_3 + \theta_4) - \theta_2 - \theta_3$$

Finally, to solve for $\theta_5$, given $\theta_1 \ldots \theta_4$, note that $\theta_4$ is now known. Therefore, we can write the following equation:

$$^0T_{4T = 0T} \Rightarrow ^4T_{5T = 0T^{-1} 5T} \Rightarrow$$

$$\begin{bmatrix}
    C_5 & -S_5 & 0 & 0 \\
    0 & 0 & -1 & -d_5 \\
    S_5 & C_5 & 0 & 0 \\
    0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
    C_1C_{234} & S_1C_{234} & S_{234} & \ldots \\
    -C_1S_{234} & -S_1S_{234} & C_{234} & \ldots \\
    S_1 & -C_1 & 0 & \ldots \\
    0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
    r_{11} & r_{12} & r_{13} & p_x \\
    r_{21} & r_{22} & r_{23} & p_y \\
    r_{31} & r_{32} & r_{33} & p_z
\end{bmatrix}
= ^0T_{4T}$$

Hence:

\begin{align*}
S_5 &= S_1r_{11} - C_1r_{21} \\
C_5 &= S_1r_{12} - C_1r_{22}
\end{align*}

$$\Rightarrow \theta_5 = A\tan 2(S_5, C_5)$$
Chapter 4 Exercises:

• 4.1, 4.2, 4.3, 4.8, 4.9
• 4.1 Programming Exercise
• 4.1 MathLab Exercise
• Programming of the PUMA 560 Inverse Kinematics