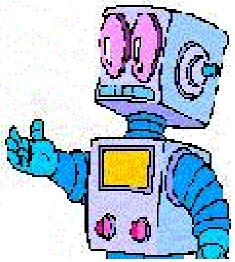
INTRODUCTION TO ROBOTICS
(Kinematics, Dynamics, and Design)

SESSION # 9:
SPATIAL DESCRIPTIONS & TRANSFORMATIONS



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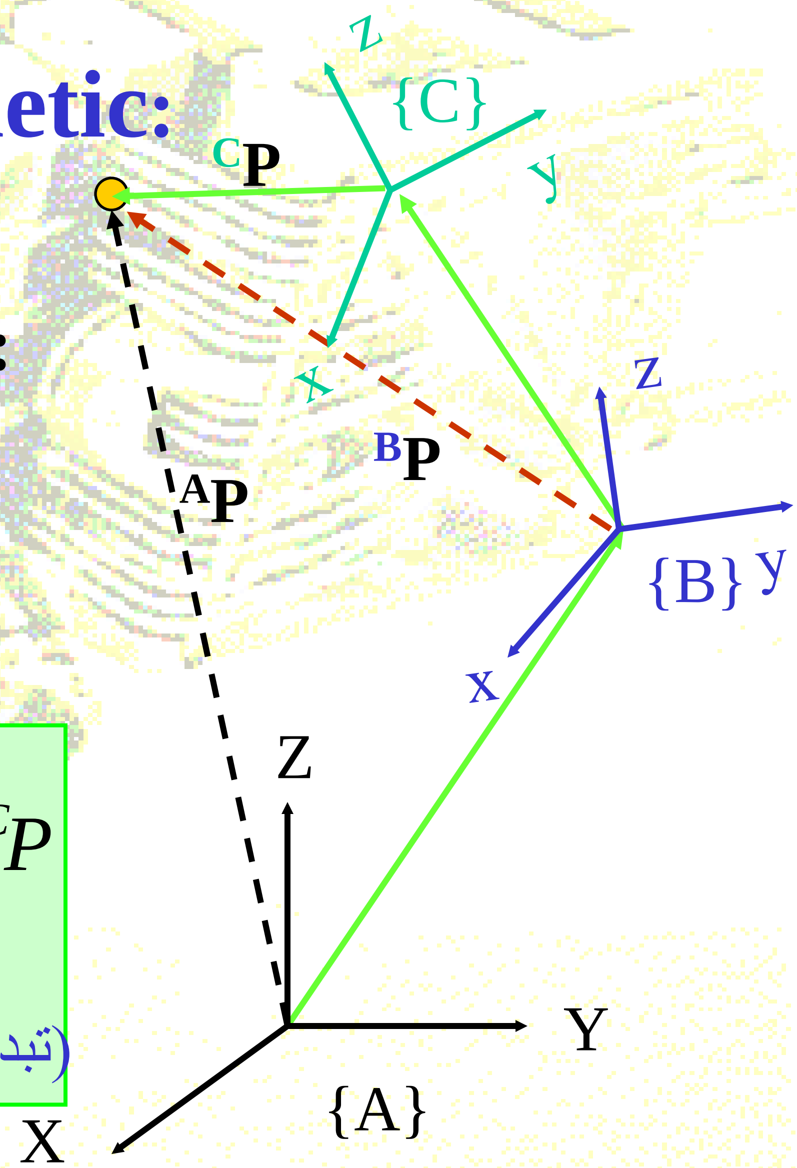
Spatial Descriptions and Transformations

- **Transformation Arithmetic:**

  - **Multiplication of Transforms (Compound Transformations):**



Given frames {A}, {B}, {C}, and vector ^{C}P:

Find: ^{A}P

We can write:

$$\begin{cases} {}^{B}P = {}^{B}_{C}T\,{}^{C}P \\ {}^{A}P = {}^{A}_{B}T\,{}^{B}P \end{cases} \Rightarrow {}^{A}P = {}^{A}_{B}T\,{}^{B}_{C}T\,{}^{C}P = {}^{A}_{C}T\,{}^{C}P$$

$$where: \quad {}^{A}_{C}T = {}^{A}_{B}T\,{}^{B}_{C}T$$ (تبدیل مرکب)

Spatial Descriptions and Transformations

• Transformation Arithmetic:

Inversion of Transforms:

One may invert the "T" matrices by standard techniques (too long). However, a simpler method exists:

Given: ${}^{A}_{B}T$

$$
{}^{A}_{B}T = \left[\begin{array}{ccc|c} & {}^{A}_{B}R & & {}^{A}P_{BORG} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] = \begin{array}{cccc} \hat{N} & \hat{O} & \hat{A} & \hat{P} \\ \downarrow & \downarrow & \downarrow & \downarrow \end{array} \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Find: ${}^{A}_{B}T^{-1} = {}^{B}_{A}T = ?$

$$
{}^{A}_{B}T^{-1} = {}^{B}_{A}T = \left[\begin{array}{ccc|c} & {}^{A}_{B}R^{T} & & -{}^{A}_{B}R^{T}\,{}^{A}P_{BORG} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] = \begin{bmatrix} n_x & n_y & n_z & -\hat{P}.\hat{N} \\ o_x & o_y & o_z & -\hat{P}.\hat{O} \\ a_x & a_y & a_z & -\hat{P}.\hat{A} \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Spatial Descriptions and Transformations

- **Example:**

Inversion of Transforms:

Given:

$$ {}^{A}_{B}T = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -0.43 & 0.87 & 0.25 & 1 \\ 0.75 & 0.5 & -0.43 & 2 \\ -0.5 & 0 & -0.87 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Find:

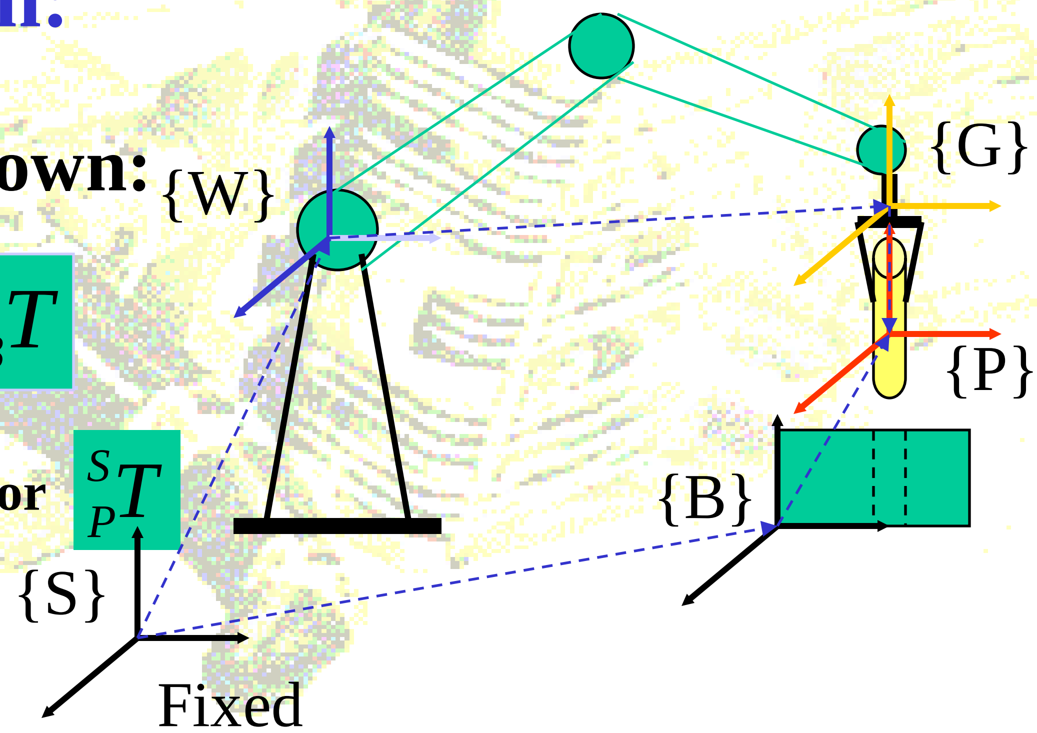
$$ {}^{A}_{B}T^{-1} = {}^{B}_{A}T = \begin{bmatrix} n_x & n_y & n_z & -\hat{P}.\hat{N} \\ o_x & o_y & o_z & -\hat{P}.\hat{O} \\ a_x & a_y & a_z & -\hat{P}.\hat{A} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -0.43 & 0.75 & -0.5 & 0.43 \\ 0.87 & 0.5 & 0 & -1.87 \\ 0.25 & -0.43 & -0.87 & 3.21 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Spatial Descriptions and Transformations

- **Transform Equation:**
  - **Consider the figure shown:**

Given: $^{S}_{W}T, {}^{W}_{G}T, {}^{G}_{P}T, {}^{B}_{P}T, {}^{S}_{B}T$

We can write two expressions for $^{S}_{P}T$



$$\begin{cases} {}^{S}_{P}T = {}^{S}_{W}T\,{}^{W}_{G}T\,{}^{G}_{P}T \\ {}^{S}_{P}T = {}^{S}_{B}T\,{}^{B}_{P}T \end{cases} \Rightarrow {}^{S}_{W}T\,{}^{W}_{G}T\,{}^{G}_{P}T = {}^{S}_{B}T\,{}^{B}_{P}T$$

Transform Equation

Spatial Descriptions and Transformations

- **Transform Equation:**

Ex: If the Transform $\boxed{{}^{B}_{P}T}$ is unknown?

We can write the following expression for the orientation of the peg at insertion:

$$\{S\} \xrightarrow{{}^{S}_{W}T} \{W\} \xrightarrow{{}^{W}_{G}T} \{G\} \xrightarrow{{}^{G}_{P}T} \{P\} \xleftarrow{{}^{B}_{P}T=?} \{B\} \xleftarrow{{}^{S}_{B}T} \{S\}$$

$$\boxed{{}^{S}_{W}T\,{}^{W}_{G}T\,{}^{G}_{P}T={}^{S}_{B}T\,{}^{B}_{P}T \Rightarrow {}^{B}_{P}T={}^{S}_{B}T^{-1}\,{}^{S}_{W}T\,{}^{W}_{G}T\,{}^{G}_{P}T}$$

Transform Equation

Spatial Descriptions and Transformations

- **More on Representation of Position & Orientation:**

Cylindrical Coordinates:

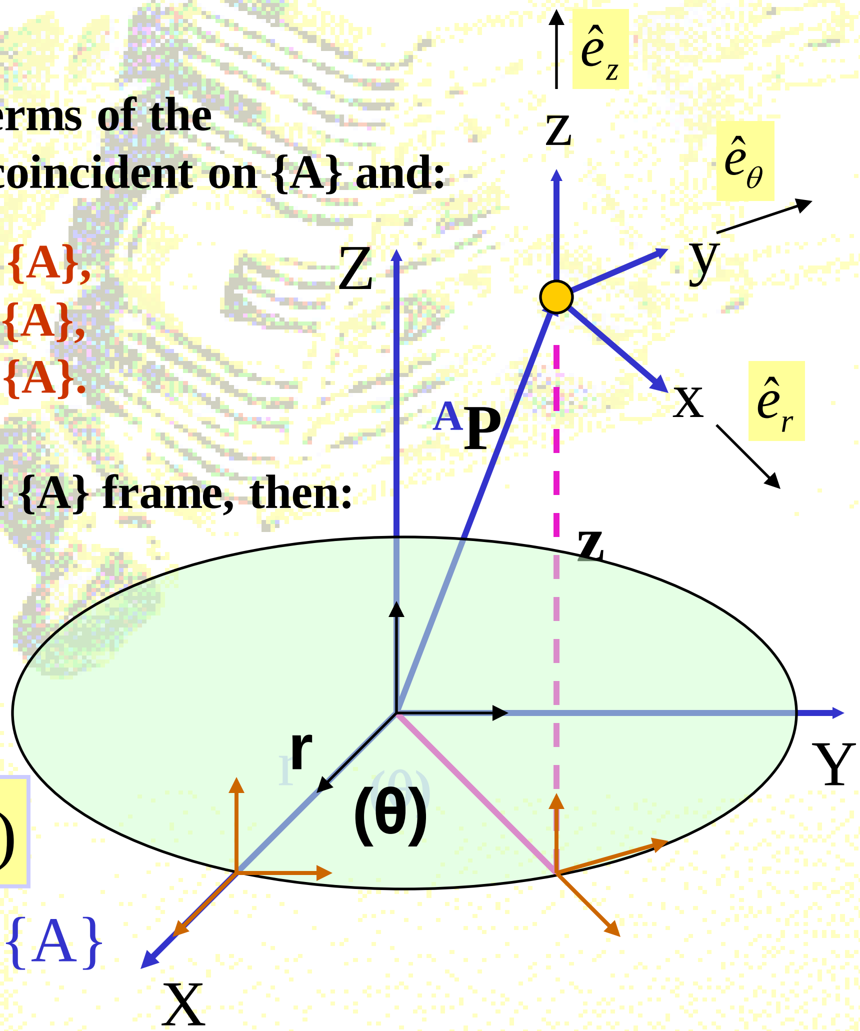
To define Cartesian coordinates of a point in terms of the Cylindrical coordinates, start by a coordinate coincident on {A} and:

1. **Translate by "r" along X-axis of the frame {A},**
2. **Rotate by an angle "θ" about the Z-axis of {A},**
3. **Translate by "z" vertically along Z-axix of {A}.**

Transformations are all along the Original/Old {A} frame, then:

***PREMULTIPLY*:**

$$T = Trans(\hat{Z}, z)Rot(\hat{Z}, \theta)Trans(\hat{X}, r)$$



Spatial Descriptions and Transformations

- **More on Representation of Position & Orientation:**
  - **Operator T:** $T = Trans(\hat{Z}, z)Rot(\hat{Z}, \theta)Trans(\hat{X}, r)$

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} Cos\theta & -Sin\theta & 0 & 0 \\ Sin\theta & Cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & r \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T = \begin{bmatrix} Cos\theta & -Sin\theta & 0 & rCos\theta \\ Sin\theta & Cos\theta & 0 & rSin\theta \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \Rightarrow \quad {}^{A}P = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} rCos\theta \\ rSin\theta \\ z \end{bmatrix}$$

Spatial Descriptions and Transformations

- **More on Representation of Position & Orientation:**

➢ **Cylindrical Coordinates:**

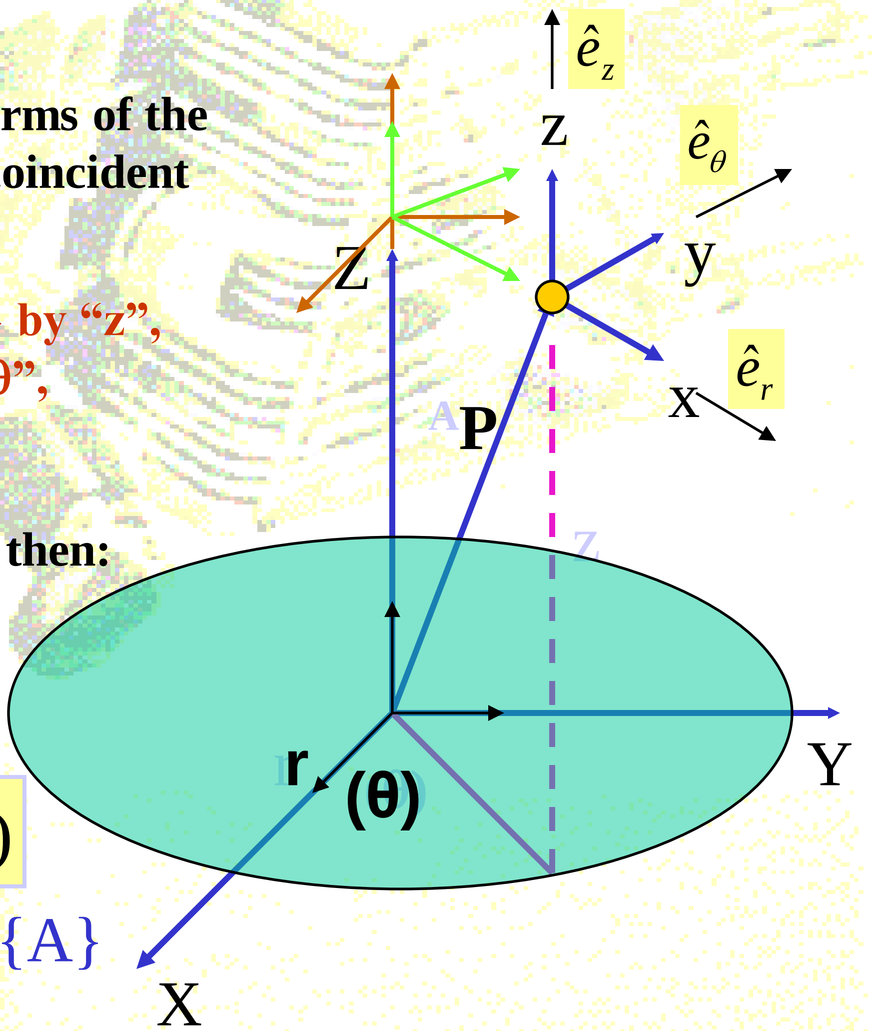
To define Cartesian coordinates of a point in terms of the Cylindrical coordinates, start by a coordinate coincident on {A} and: (Another Approach):

1. **Translate along the Z-axis of the frame {A} by "z",**
2. **Rotate about the New Z-axis by an angle "θ",**
3. **Translate along the New X-axis by "r".**

Transformations are all along the New frames, then:

POSTMULTIPLY:

$$T = Trans(\hat{Z}, z)Rot(\hat{Z}, \theta)Trans(\hat{X}, r)$$



Spatial Descriptions and Transformations

- Right-to-Left (Pre-Multiply) vs. Left-to-Right (Post-Multiply):

> **Example:**
> 1. Rotate 30^0 about X-axis,
> 2. Rotate 90^0 about the New (transformed) Y-axis,
> 3. Translate 3" along the Old (fixed) Z-axis,
> 4. Rotate 30^0 about the New (transformed) X-axis.

To write the corresponding transform expression, Just Remember:

{Fixed(Old) on the Left}, and {New(Transformed) on the Right}.

Therefore, the 1st transform is:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & Cos30 & -Sin30 & 0 \\ 0 & Sin30 & Cos30 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The 2nd transform is: (New-Right)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & Cos30 & -Sin30 & 0 \\ 0 & Sin30 & Cos30 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Spatial Descriptions and Transformations

- Right-to-Left (Pre-Multiply) vs. Left-to-Right (Post-Multiply):
  - **Example:** 1. Rotate 30^0 about X-axis,
    2. Rotate 90^0 about the New (transformed) Y-axis,
    3. Translate 3" along the Old (fixed) Z-axis,
    4. Rotate 30^0 about the New (transformed) X-axis.

The 3rd transform is: (Old-Left)

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&3\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&Cos30&-Sin30&0\\0&Sin30&Cos30&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}0&0&1&0\\0&1&0&0\\-1&0&0&0\\0&0&0&1\end{bmatrix}$$

The 4th transform is: (New-Right)

$$T=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&3\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&Cos30&-Sin30&0\\0&Sin30&Cos30&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}0&0&1&0\\0&1&0&0\\-1&0&0&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&Cos30&-Sin30&0\\0&Sin30&Cos30&0\\0&0&0&1\end{bmatrix}$$

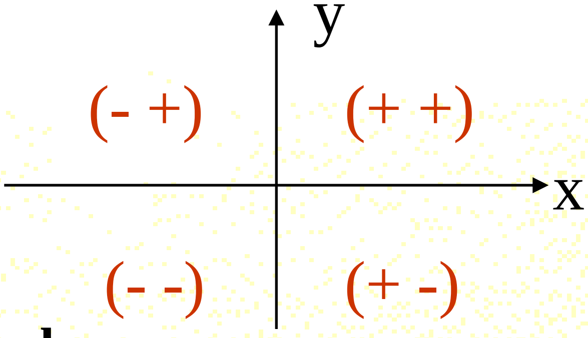
Spatial Descriptions and Transformations

- **More on Representation of Position & Orientation :**

To find Cylindrical coordinates from Cartesian Coordinates:

$$^{A}P = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} rCos\theta \\ rSin\theta \\ z \end{bmatrix} \Rightarrow {}^{A}P = \begin{bmatrix} r \\ \theta \\ z \end{bmatrix} = \begin{bmatrix} \sqrt{p_x^2 + p_y^2} \\ Atan2(p_y, p_x) \\ p_z \end{bmatrix}$$

- **Atant2(p_y, p_x): is a "Two-Argument" arc tangent function. It computes $\tan^{-1}(p_y/p_x)$, but uses the signs of both p_y and p_x to determine the quadrant in which the resulting angle lies.**



Ex: Atan2(y,x)=$\tan^{-1}$(y/x)=Atan2(-2,-2) = -135^0
Atan2(2,2) = 45^0
Atan2(-2,2) = -45^0
Atan2(2,-2) = 135^0
Atan2(0,0) = Undefined

Spatial Descriptions and Transformations

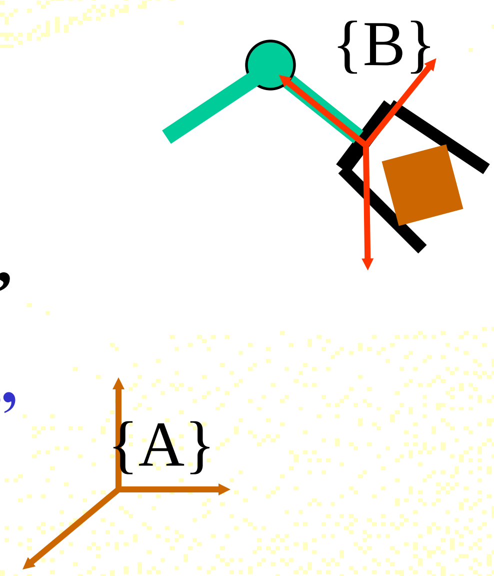
- **More on Representation of Orientation:**
  - **So far we introduced a (3×3) Rotation Matrix to define orientation, such that:**

$$
{}^{A}_{B}R = \begin{bmatrix} {}^{A}\hat{X}_{B} & {}^{A}\hat{Y}_{B} & {}^{A}\hat{Z}_{B} \end{bmatrix}
$$

$$
\begin{cases} \left|\hat{X}\right| = 1, \quad \left|\hat{Y}\right| = 1, \quad \left|\hat{Z}\right| = 1 \\ \hat{X}\cdot\hat{Y} = 0, \quad \hat{X}\cdot\hat{Z} = 0, \quad \hat{Y}\cdot\hat{Z} = 0 \end{cases}
$$

9-Quantities (کمیت–9),
and 6-Dependencies (قید–6)

  - **To specify the desired orientation of a robot hand, it is difficult to input a nine-element matrix with orthogonal columns. Therefore, we need:**
    "A more efficient way to specify orientation"
    Several methods are present.

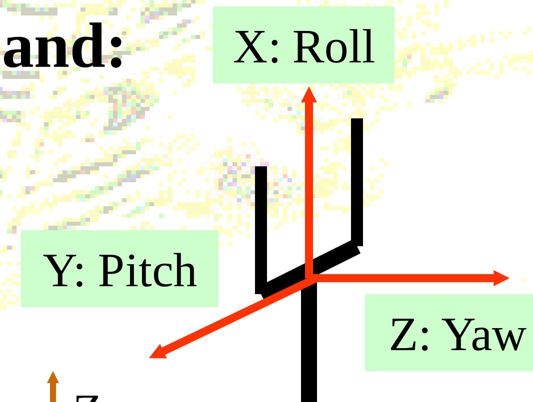


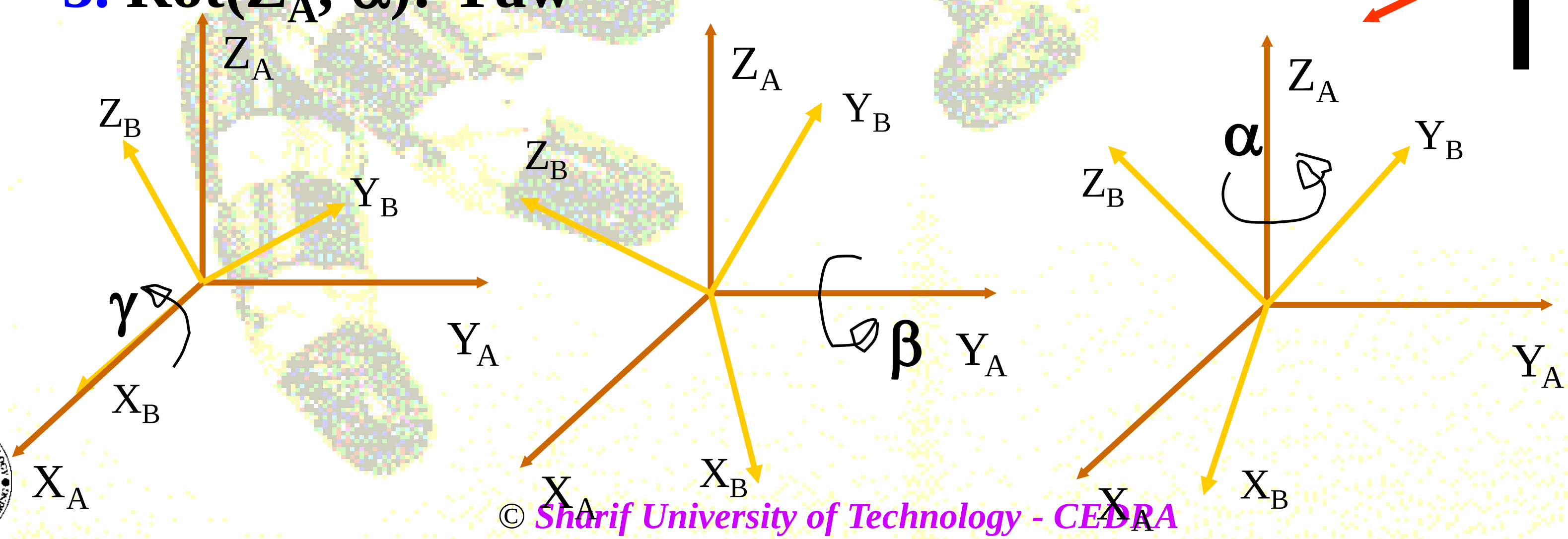
Spatial Descriptions and Transformations

- **More on Representation of Orientation:**
  - **Roll, Pitch, and Yaw (Fixed) Angles about <u>Fixed axes</u> (RPY):**

To describe orientation of {B} relative to a fixed known frame {A}, start with a frame coincident with {A} and:

1. **Rot(X_A, γ): Roll**
2. **Rot(Y_A, β): Pitch**
3. **Rot(Z_A, α): Yaw**





Spatial Descriptions and Transformations

- **More on Representation of Orientation:**
  - **Since all rotations are about the original/fixed frame {A}, then Pre-multiply to find the RPY-Operator as:**

$$
{}^{A}_{B}R_{rpy}(\gamma,\beta,\alpha)=Rot(\hat{Z}_A,\alpha)Rot(\hat{Y}_A,\beta)Rot(\hat{X}_A,\gamma)
$$

$$
=\begin{bmatrix} C\alpha & -S\alpha & 0 \\ S\alpha & C\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} C\beta & 0 & S\beta \\ 0 & 1 & 0 \\ -S\beta & 0 & C\beta \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & C\gamma & -S\gamma \\ 0 & S\gamma & C\gamma \end{bmatrix}=
$$

$$
=\begin{bmatrix} C\alpha C\beta & C\alpha S\beta S\gamma - S\alpha C\gamma & C\alpha S\beta C\gamma + S\alpha S\gamma \\ S\alpha C\beta & S\alpha S\beta S\gamma + C\alpha C\gamma & S\alpha S\beta C\gamma - C\alpha S\gamma \\ -S\beta & C\beta S\gamma & C\beta C\gamma \end{bmatrix}
$$

Spatial Descriptions and Transformations

- **More on Representation of Orientation:**
  - **Inverse of this problem is to compute the Roll, Pitch, and Yaw angles for a given Rotation Matrix:**

$$
{}^{A}_{B}R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} 9-Equations \\ 3-Unknowns \\ 6-Dependencies \end{bmatrix} =
$$

$$
{}^{A}_{B}R_{rpy}(\gamma,\beta,\alpha) = \begin{bmatrix} C\alpha C\beta & C\alpha S\beta S\gamma - S\alpha C\gamma & C\alpha S\beta C\gamma + S\alpha S\gamma \\ S\alpha C\beta & S\alpha S\beta S\gamma + C\alpha C\gamma & S\alpha S\beta C\gamma - C\alpha S\gamma \\ -S\beta & C\beta S\gamma & C\beta C\gamma \end{bmatrix}
$$

Spatial Descriptions and Transformations

- **More on Representation of Orientation:**
  - **Therefore with 3-independent equations, one can find the 3-unknowns Roll, Pitch, and Yaw angles as:**

$$\gamma = A\tan 2(r_{32}/C\beta, r_{33}/C\beta)$$

$$\beta = A\tan 2(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2})$$

$$\alpha = A\tan 2(r_{21}/C\beta, r_{11}/C\beta)$$

(as long as $C\beta \neq 0$)

Read the detailed discussion of the solution in your book.

Spatial Descriptions and Transformations

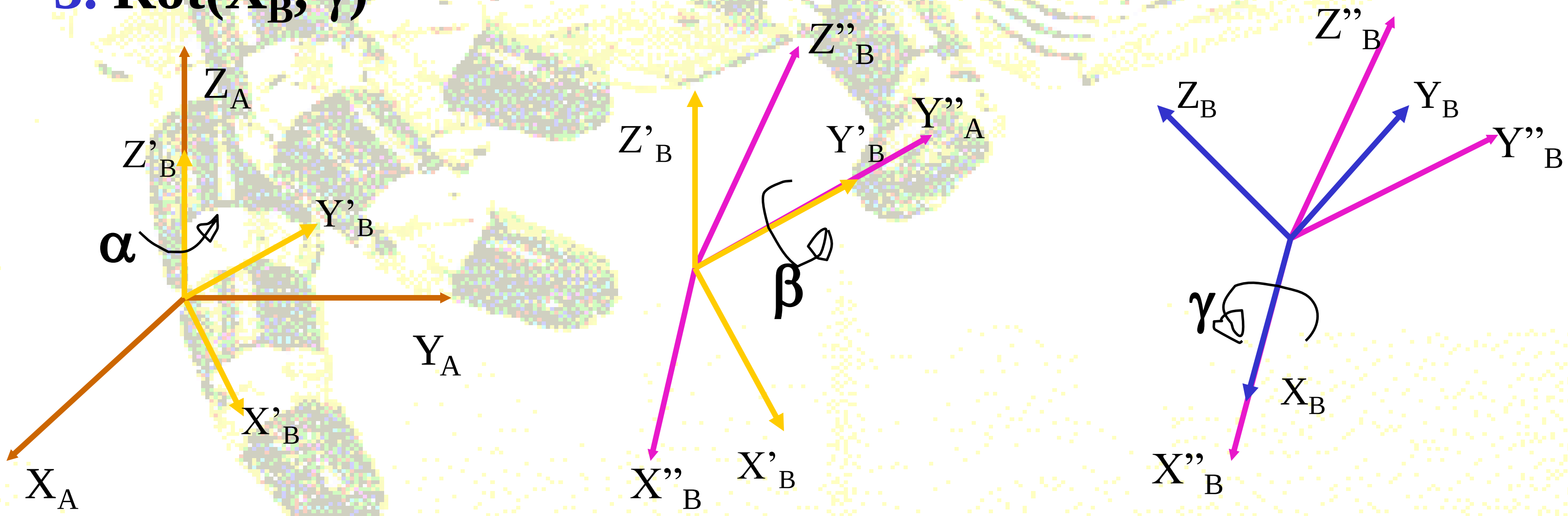
- **More on Representation of Orientation:**

➢ **Euler Angles about Moving axes:** Another method to represent orientation.

(Z-Y-X) Euler Angles:

To describe orientation of {B} relative to a fixed known frame {A}, start with a frame coincident with {A} and:

1. $Rot(Z_B, \alpha)$
2. $Rot(Y_B, \beta)$
3. $Rot(X_B, \gamma)$



Spatial Descriptions and Transformations

- **More on Representation of Orientation:**
  - **Since all rotations are about the Moving/New frame {B}, then Post-multiply to find the Euler-Operator as (same result as RPY):**

$$
{}^{A}_{B}R_{zyx}(\alpha,\beta,\gamma)=Rot(\hat{Z}_B,\alpha)Rot(\hat{Y}_B,\beta)Rot(\hat{X}_B,\gamma)
$$

$$
=\begin{bmatrix} C\alpha & -S\alpha & 0 \\ S\alpha & C\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} C\beta & 0 & S\beta \\ 0 & 1 & 0 \\ -S\beta & 0 & C\beta \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & C\gamma & -S\gamma \\ 0 & S\gamma & C\gamma \end{bmatrix}=
$$

$$
=\begin{bmatrix} C\alpha C\beta & C\alpha S\beta S\gamma - S\alpha C\gamma & C\alpha S\beta C\gamma + S\alpha S\gamma \\ S\alpha C\beta & S\alpha S\beta S\gamma + C\alpha C\gamma & S\alpha S\beta C\gamma - C\alpha S\gamma \\ -S\beta & C\beta S\gamma & C\beta C\gamma \end{bmatrix}
$$

Spatial Descriptions and Transformations

- **More on Representation of Orientation:**

  - **Simple (General) Rotation:** **Rotation of a rigid body (frame) about a general fixed axis in space.**

  - **Elementary Rotation:** **Rotation of a rigid body (frame) about one of the coordinate axes.**

  - **Euler's Theorem:** **Any change of orientation (about an arbitrary axis) for a rigid body with a fixed body point can be accomplished through a *simple rotation*. The rigid body rotation can be resolved into three *elementary rotations*, where the angles of these rotations are called the Euer's Angles.**

(The 3-independent Eulerian Angles and the Fixed Angles conventions may be selected in a variety of ways and sequences. A total of 24 conventions exist of which only 12 sets are unique (see pages489-491).

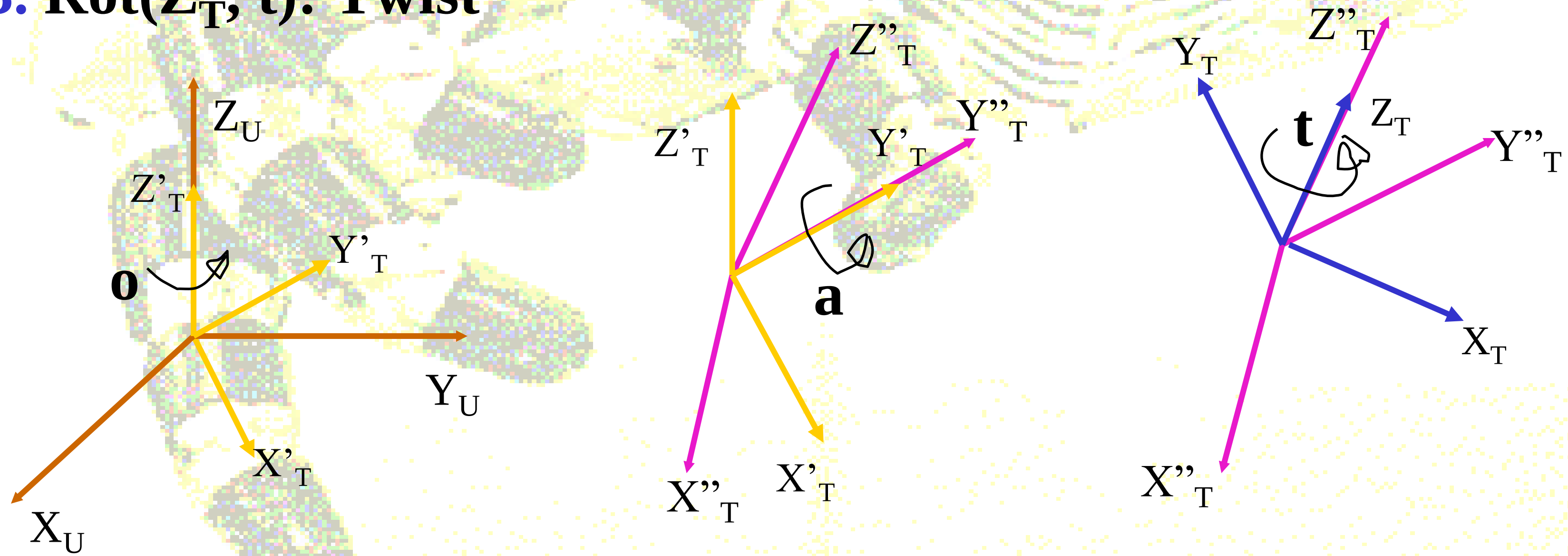
Spatial Descriptions and Transformations

- **Example:**

  - **The Unimation PUMA 560 Euler Angles Convention:**

Description of orientation of the Tool Frame {T} relative to the fixed Universal frame {U}:

1. **Rot(Z_U, o): Orientation**
2. **Rot(Y_T, a): Approach**
3. **Rot(Z_T, t): Twist**

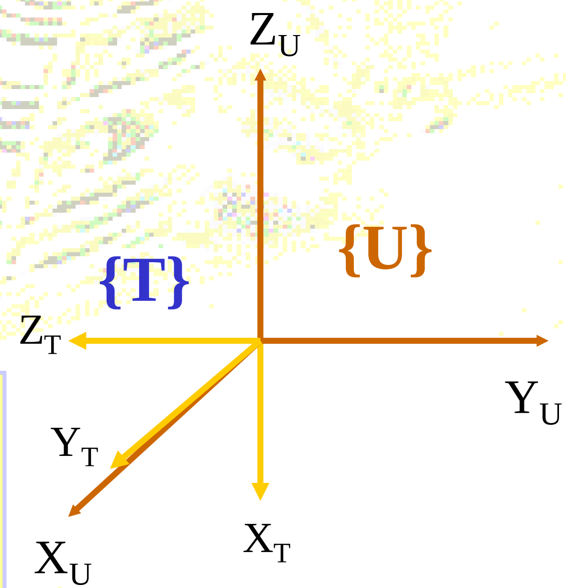


Spatial Descriptions and Transformations

- **The Unimation PUMA 560 Euler Angles Convention:**

For PUMA-560 the Tool Frame {T} is not coincident with the Universal frame {U}. Therefore, the "Zero" orientation of {T} is:

$$ {}^{U}_{T}R_{initial} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix} $$



$$ {}^{U}_{T}R = Rot(\hat{Z},o)\,{}^{U}_{T}R_{initial}\,Rot(\hat{Y},a)\,Rot(\hat{Z},t) = $$

$$ = \begin{bmatrix} -SoSaCt + CoSt & -SoSaSt + CoCt & SoCa \\ CoSaCt + SoSt & -CoSaSt + SoCt & -CoCa \\ -CaCt & CaSt & -Sa \end{bmatrix} $$

Spatial Descriptions and Transformations

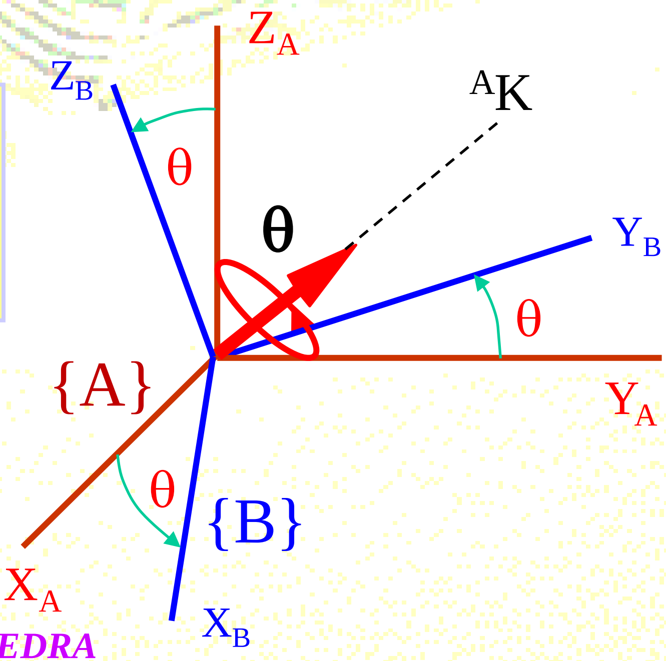
- **Equivalent Angle-Axis Representation (Euler's Theorem on Rotation): Any orientation of a rigid body (frame) can be obtained through a proper Axis and Angle selection.**

➢ **A Simple (General) Rotation Operator = Rot(^{A}K, θ):**

Rotation of a rigid body (frame) about a general fixed axis "^{A}K" in space.

Originally {B} is coincident with {A}, then applying Rot(^{A}K, θ) by Right-Hand-Rule, we can define:

$$ {}^{A}_{B}R({}^{A}\hat{K},\theta)=\begin{bmatrix} k_x k_x v\theta + C\theta & k_x k_y v\theta - k_z S\theta & k_x k_z v\theta + k_y S\theta \\ k_x k_y v\theta + k_z S\theta & k_y k_y v\theta + C\theta & k_y k_z v\theta - k_x S\theta \\ k_x k_z v\theta - k_y S\theta & k_y k_z v\theta + k_x S\theta & k_z k_z v\theta + C\theta \end{bmatrix} $$



where:

$$ {}^{A}\hat{K} = k_x \underline{i} + k_y \underline{j} + k_z \underline{k} $$

$$ v\theta = 1 - Cos\theta $$

$$ k_x^2 + k_y^2 + k_z^2 = 1 $$

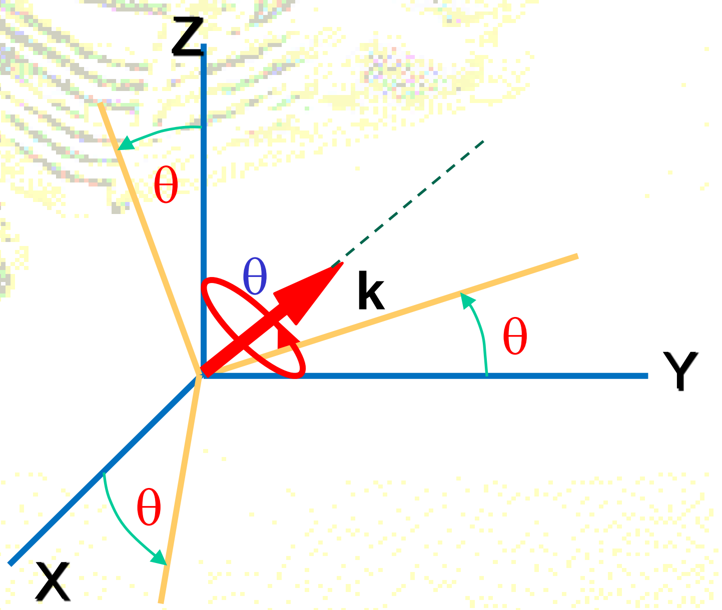
Spatial Descriptions and Transformations

- **Equivalent Angle-Axis Representation (Euler's Theorem on Rotation):** **When the axis of rotation is chosen as one of the principal axes of {A}, then the Equivalent (General) Rotation Matrix take on the familiar form of Planar (Elementary) Rotations:**

$$Rot({}^A\hat{X},\theta)=\begin{bmatrix}1 & 0 & 0\\ 0 & Cos\theta & -Sin\theta\\ 0 & Sin\theta & Cos\theta\end{bmatrix}, \Leftrightarrow k_x=1, k_y=0, k_z=0$$

$$Rot({}^A\hat{Y},\theta)=\begin{bmatrix}Cos\theta & 0 & Sin\theta\\ 0 & 1 & 0\\ -Sin\theta & 0 & Cos\theta\end{bmatrix}, \Leftrightarrow k_x=0, k_y=1, k_z=0$$

$$Rot({}^A\hat{Z},\theta)=\begin{bmatrix}Cos\theta & -Sin\theta & 0\\ Sin\theta & Cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}, \Leftrightarrow k_x=0, k_y=0, k_z=1$$



Spatial Descriptions and Transformations

- **Equivalent Angle-Axis Representation (Euler's Theorem on Rotation): To obtain (^{A}K, θ) from a given rotation matrix (orientation):**

$$ {}^{A}_{B}R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = {}^{A}_{B}R({}^{A}\hat{K}, \theta) \quad \Rightarrow $$

$$ Sin\theta = \pm \frac{1}{2} \sqrt{(r_{32} - r_{23})^2 + (r_{13} - r_{31})^2 + (r_{21} - r_{12})^2} $$

$$ Cos\theta = \left(\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right) \quad \Rightarrow \theta = A\tan 2\left(\frac{Sin\theta}{Cos\theta}\right) $$

$$ \hat{K} = \frac{1}{2Sin\theta} \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix} = \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix} $$

Spatial Descriptions and Transformations

- **Any combination of rotations is always equivalent to a single rotation about some axis "K" by an angle "θ":**

➢ **Example:**

Consider the following combined rotation operators, and obtain its corresponding equivalent angle-axis representation?

$$Rot(\hat{Y},90)Rot(\hat{Z},90)=\begin{bmatrix}0&0&1&0\\1&0&0&0\\0&1&0&0\\0&0&0&1\end{bmatrix}\Rightarrow\left\{\begin{matrix}Sin\theta=\pm\sqrt{(1-0)^2+(1-0)^2+(1-0)^2}=\pm\dfrac{\sqrt{3}}{2}\\ Cos\theta=(\dfrac{0+0+0-1}{2})=-\dfrac{1}{2}\end{matrix}\right\}\Rightarrow$$

$$\theta=A\tan 2(\pm\frac{\sqrt{3}}{2}\Big/-\frac{1}{2})=\pm 120^{\circ},\qquad \hat{K}=\pm(\frac{1}{\sqrt{3}}\underline{i}+\frac{1}{\sqrt{3}}\underline{j}+\frac{1}{\sqrt{3}}\underline{k})$$

$$Rot(\hat{Y},90)Rot(\hat{Z},90)\equiv Rot(\hat{K},\pm 120)$$

Spatial Descriptions and Transformations

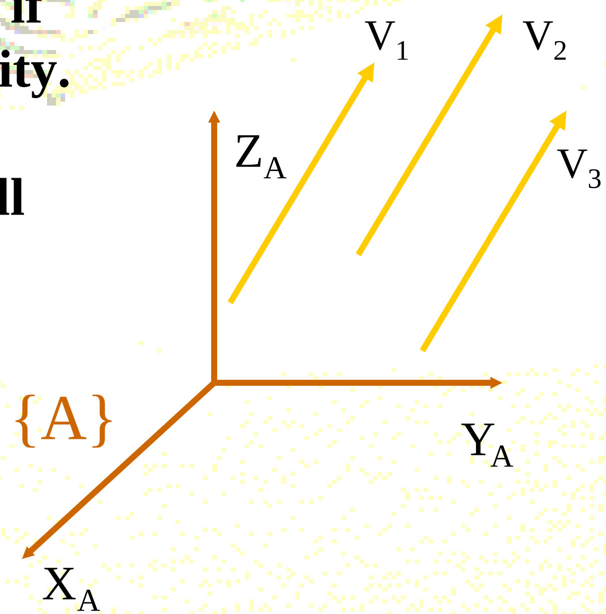
- **Transformations of Free and Line Vectors:**

In mechanics we make a distinction between the equality and the equivalence of vectors.

- Two vectors are equal if they have the same dimensions, magnitude, and direction.

- Two equal vectors may have different lines of actions. (Ex. Velocity vectors shown).

- Two vectors are equivalent in a certain capacity if each produces the very same effect in this capacity.



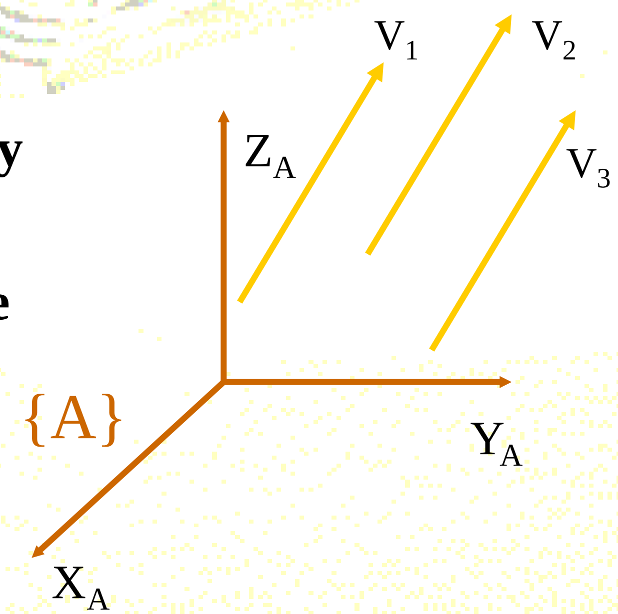
*** If the criterion in this Ex. is distance traveled, all three vectors give the same result and are thus equivalent in this sense.**

*** If the criterion in this Ex. is height above the XY-plane, then the vectors are not equivalent despite their equality.**

Spatial Descriptions and Transformations

- **Transformations of Free and Line Vectors:**
  - **A Line-Vector (بردار خطی):** A vector which, along with direction and magnitude, is also dependent on its line-of-action (or point-of-action) as far as determining its effects is concerned. (Ex: A force vector, A position vector).
  - **A Free-Vector (بردار آزاد):** A vector which may be positioned anywhere in space without loss or change of meaning provided that magnitude and direction are preserved. (Ex: A pure moment vector, A velocity vector).

Therefore, in transforming free vectors from one frame to another frame, only the rotation matrix relating the two frames is used.



$$\boxed{{}^{A}V = {}^{A}_{B}R\,{}^{B}V \qquad \textit{and not} \qquad \cancel{{}^{A}V = {}^{A}_{B}T\,{}^{B}V}}$$

Spatial Descriptions and Transformations

- **Computational Considerations:**

Efficiency in computing methods is an important issue in Robotics.

➢ **Example: Consider the following transformations:**

1st Approach:

$$^{A}P = \underbrace{({}^{A}_{B}R\,{}^{B}_{C}R\,{}^{C}_{D}R)}_{(54Mul.+36Add.)}{}^{D}P = \underbrace{{}^{A}_{D}R\,{}^{D}P}_{(9Mul.+6Add.)} \Rightarrow \quad (63Mul.+42Add.)$$

2nd Approach:

$$^{A}P = {}^{A}_{B}R\,{}^{B}_{C}R\,{}^{C}_{D}R\,{}^{D}P = {}^{A}_{B}R\,{}^{B}_{C}R\,{}^{C}P = {}^{A}_{B}R\,{}^{B}P \Rightarrow \quad (27Mul.+18Add.)$$

*** The 2nd Approach is more efficient. ***.