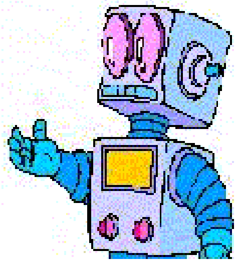
INTRODUCTION TO ROBOTICS

(Kinematics, Dynamics, and Design)

SESSION # 8:

SPATIAL DESCRIPTIONS & TRANSFORMATIONS



Ali Meghdari, Professor

School of Mechanical Engineering

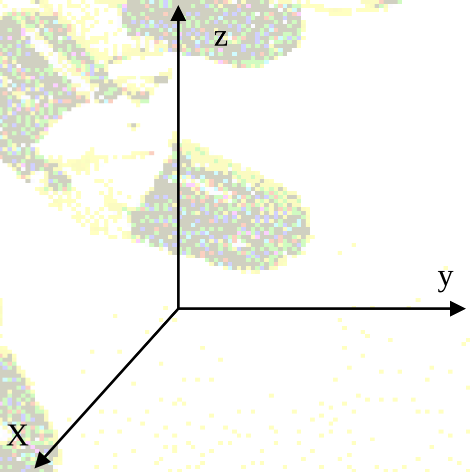
Sharif University of Technology

Tehran, IRAN 11365-9567

Homepage: http://meghdari.sharif.edu

Spatial Descriptions

- ***Reference Frame***
  - A Cartesian coordinate system with ***3 orthogonal axes***
  - Frames may be specified with respect to other frames
  - Spherical and cylindrical systems can also be used – we will use Cartesian coordinate systems.



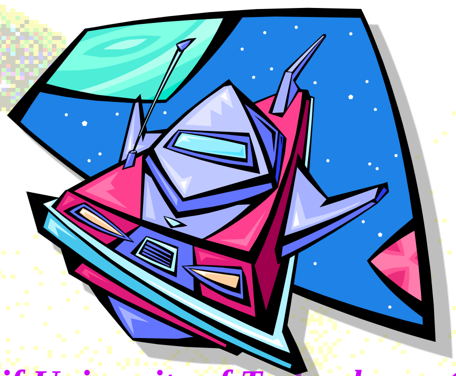
Degrees of Freedom

How many ***degrees of freedom*** does a "***point***" has in 3 space?

Three: can move in x, y, and z directions.

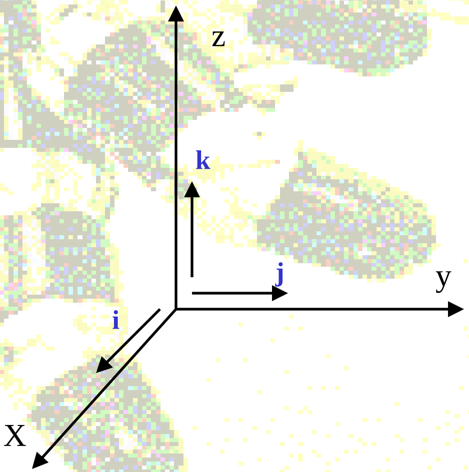
How many ***degrees of freedom*** does a 3D Object has in 3 space?

Six: can move in x, y, z directions and rotate around those axes.



Points

- ***Position***
  - Can be specified in 3 space by a ***3x1*** position vector
  - The vector is specified with respect to some ***reference frame***
  - ***Unit vectors*** are vectors of length/magnitude 1
  - In this presentation i, j, and k are unit vectors oriented along the x, y and z axes respectively



$$P = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

Understood to be a vector defined as

Xi + Yj + Zk

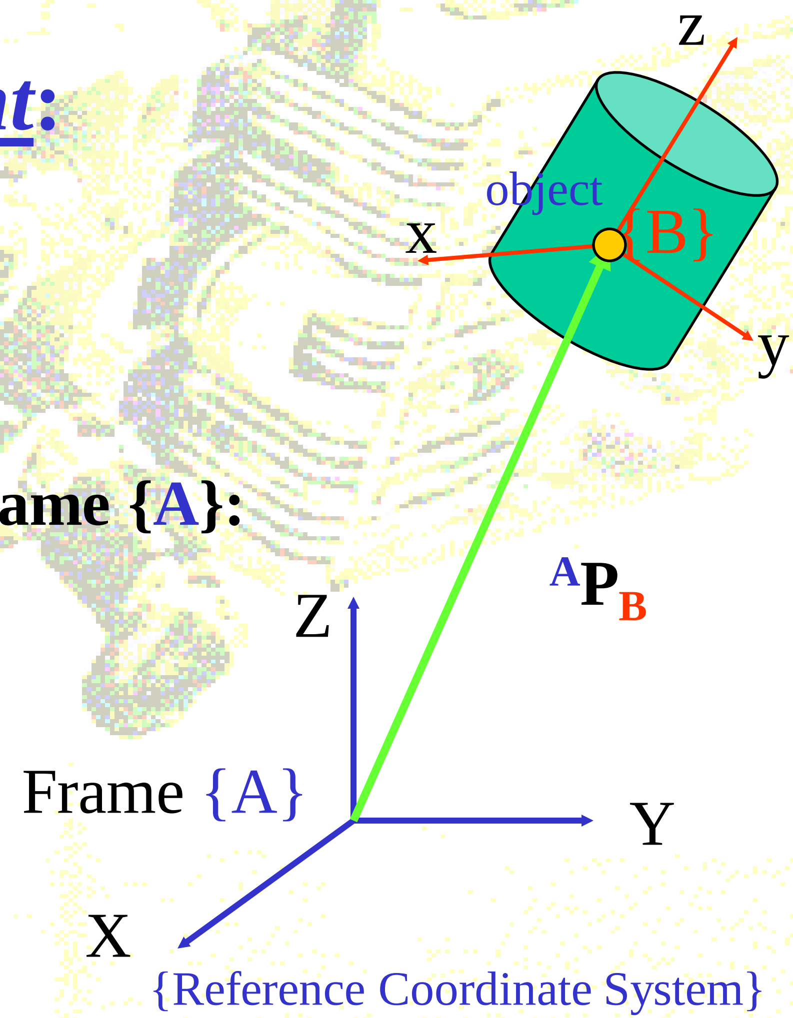
Spatial Descriptions and Transformations

- **Description of a *Point*:**

Position Vector

- **Position of Point B in Frame {A}:**

$$ {}^{A}P_{B} = [P_x \;\; P_y \;\; P_z]^T $$

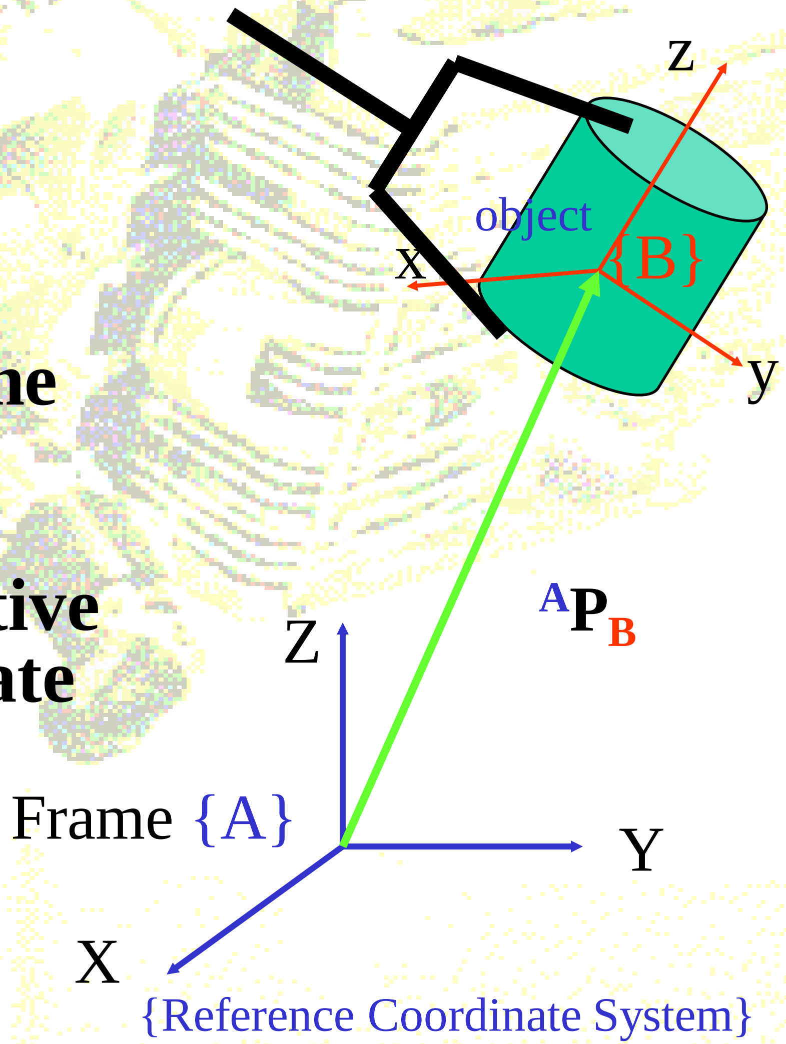


Spatial Descriptions and Transformations

- **Description of an *Orientation* (جهت)**

> **Attach a coordinate frame to the body, then**

> **Describe this frame relative to the reference coordinate system.**



Spatial Descriptions and Transformations

- **Express Unit Vectors of {B} in terms of the {A} system.**

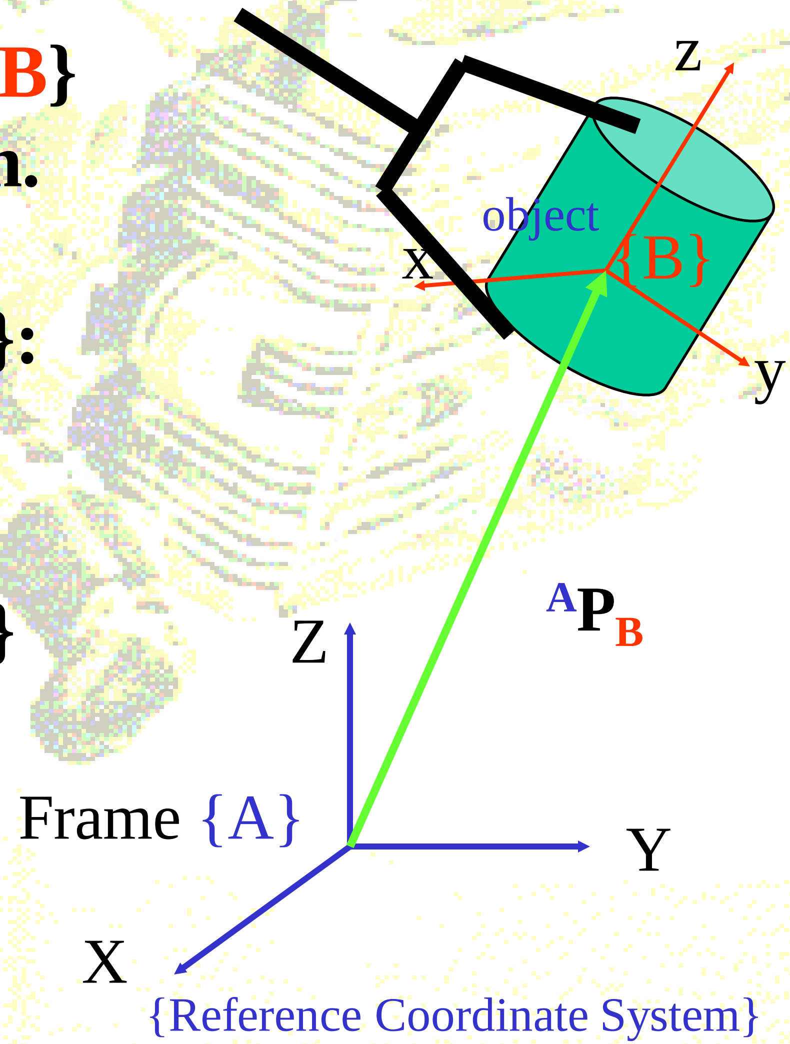
➢ **Unit Vectors of frame {B}:**

$$\hat{X}_B, \hat{Y}_B, \hat{Z}_B$$

➢ **Unit Vectors of frame {B} expressed relative to {A}:**

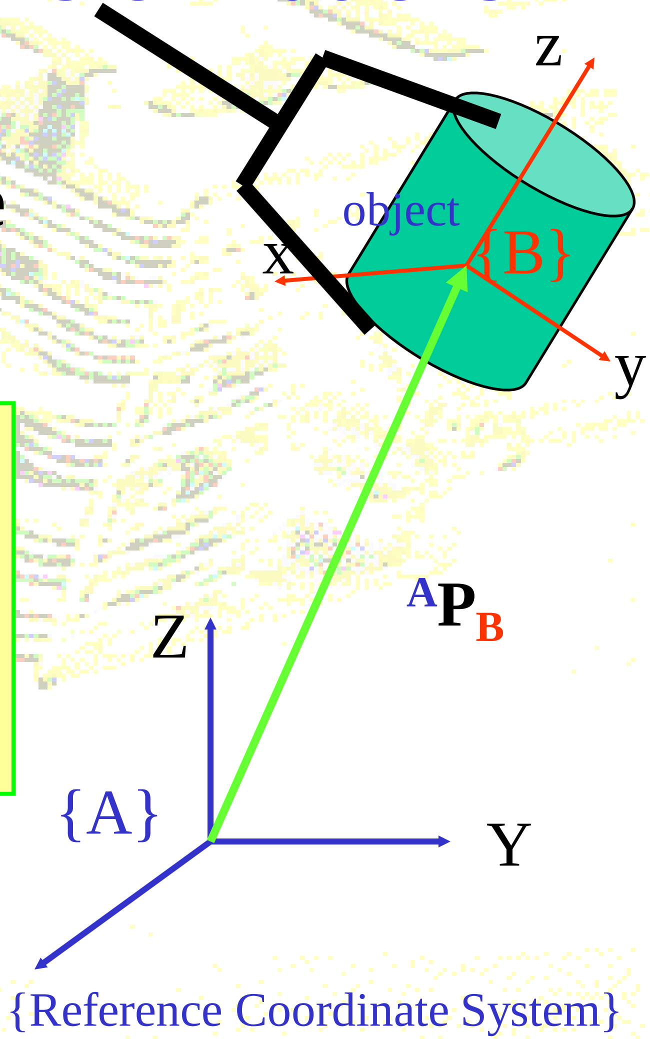
$$^{A}\hat{X}_B, {}^{A}\hat{Y}_B, {}^{A}\hat{Z}_B$$

(each having 3-components)



Spatial Descriptions and Transformations

- **Compose a *[Rotation Matrix]* expressing orientation of the frame {B} relative to {A}.**



$$[{}^{A}\hat{X}_{B}, {}^{A}\hat{Y}_{B}, {}^{A}\hat{Z}_{B}] = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = {}^{A}_{B}R$$

- **All columns have unit magnitude,**
- **Dot-Product of any two columns is zero,**
- **Hence, Rotation Matrix is *Orthogonal*.**

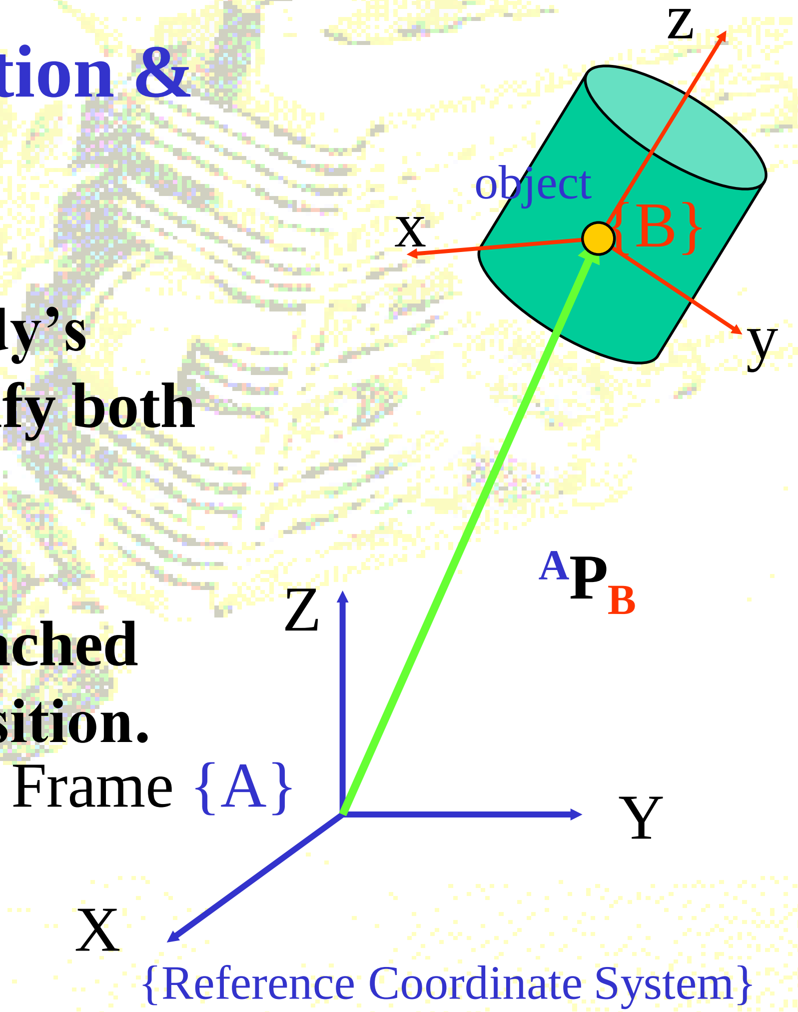
Spatial Descriptions and Transformations

- **Description of a *Frame* (Position & Orientation)**

  - **To completely specify a Rigid Body's location in space, we need to specify both position & orientation.**

  - **Choose the origin of the body-attached frame to describe rigid body's position.**

$$\{B\} = \{{}^{A}_{B}R_{(3\times 3)}, {}^{A}P_{BORG\,(3\times 1)}\}$$



${}^{A}P_{BORG}$ **: Position of origin of {B} with respect to {A}**

Spatial Descriptions and Transformations

- **Description of a *Position* (مكان)**

$$\{B\} = \begin{bmatrix} 1 & 0 & 0 & P_x \\ 0 & 1 & 0 & P_y \\ 0 & 0 & 1 & P_z \end{bmatrix} = \left[I_{(3\times3)}, {}^{A}P_{BORG} \right]$$

- **Description of an *Orientation* (جهت)**

$$\{B\} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \end{bmatrix} = \left[{}^{A}_{B}R, Zero-Vector \right]$$

Spatial Descriptions and Transformations

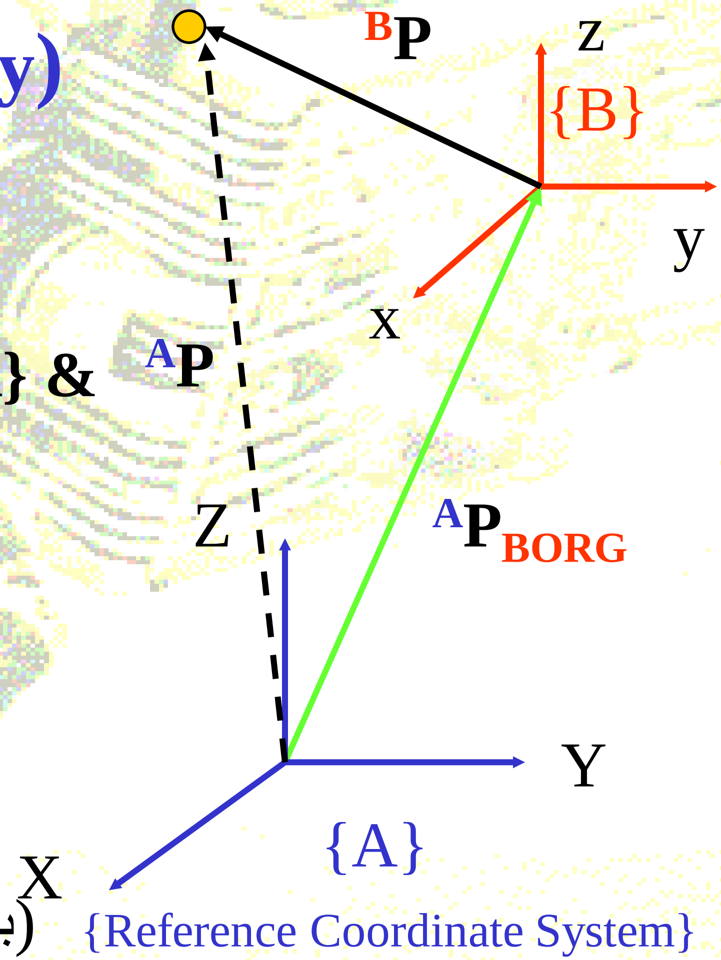
- ***Mapping*** **(Changing Relativity) (نگاشت- تغییرنسبیت):**

➢ ***Translated Frames*****: (frames {A} & {B} having the same orientation)**

Given: ^{B}P, Find: ^{A}P

$$^{A}P = {}^{B}P + {}^{A}P_{BORG}$$

$^{A}P_{BORG}$: Translation Vector (بردار انتقال)



Note: You can only add two vectors when they are expressed in frames with the same orientation.

Spatial Descriptions and Transformations

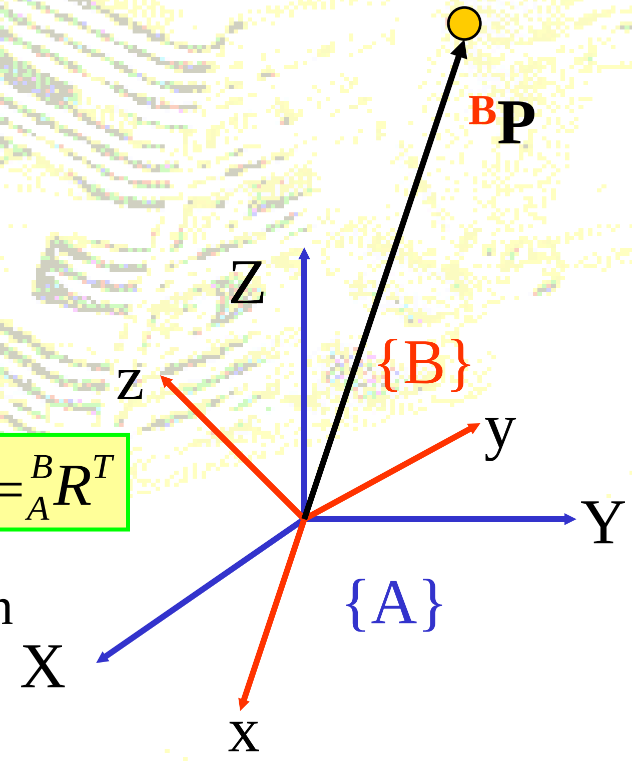
- ***Mapping*** **(Changing Relativity)** **(نگاشت- تغییرنسبیت):**

➢ ***Rotated Frames:***

Given: ${}^{B}P$, Find: ${}^{A}P$

$${}^{A}_{B}R = [{}^{A}\hat{X}_{B}\ {}^{A}\hat{Y}_{B}\ {}^{A}\hat{Z}_{B}] \equiv Orthogonal \Rightarrow {}^{A}_{B}R = {}^{B}_{A}R^{-1} = {}^{B}_{A}R^{T}$$

Columns of ${}^{A}_{B}R$ are the unit vectors of {B} written in {A} frame.
Rows of ${}^{A}_{B}R$ are the unit vectors of {A} written in {B} frame.

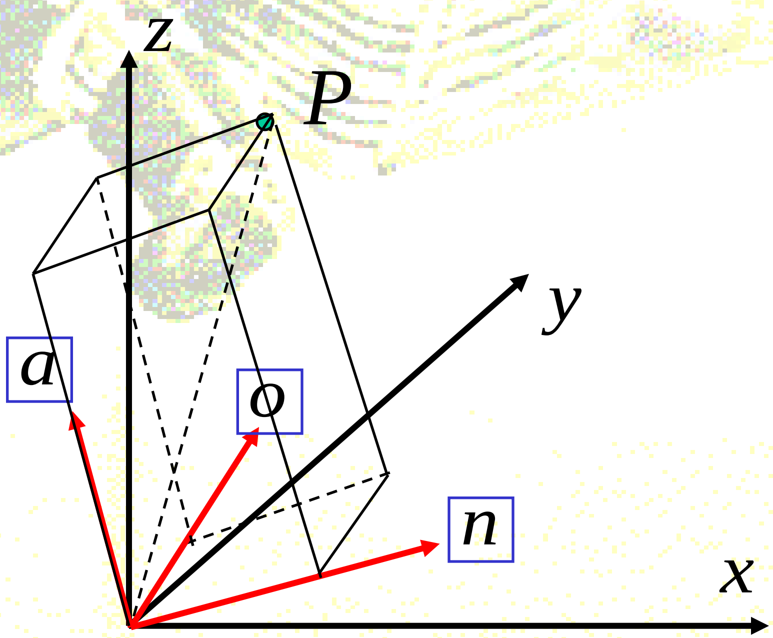


$${}^{A}P = {}^{A}_{B}R\,{}^{B}P$$

Frames inside of Frames

- How to represent ***a frame*** with respect to ***another frame*** when the origins are coincident?
  - Frames represented by 3 orthogonal unit vectors
  - Sometimes known as normal, orientation, and approach
  - Each represented with respect to the reference frame

$$R = \begin{bmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{bmatrix}$$



Dot Product

Let **x** and **y** be arbitrary vectors in 3-space and let theta be the angle between them.

$$x \cdot y = |x||y| \cos\theta$$

Orthogonality requires that

$$\vec{n} \cdot \vec{o} = 0$$

$$\vec{n} \cdot \vec{a} = 0$$

$$\vec{o} \cdot \vec{a} = 0$$

Unit vectors means that

$$|\vec{n}| = 1$$

$$|\vec{o}| = 1$$

$$|\vec{a}| = 1$$

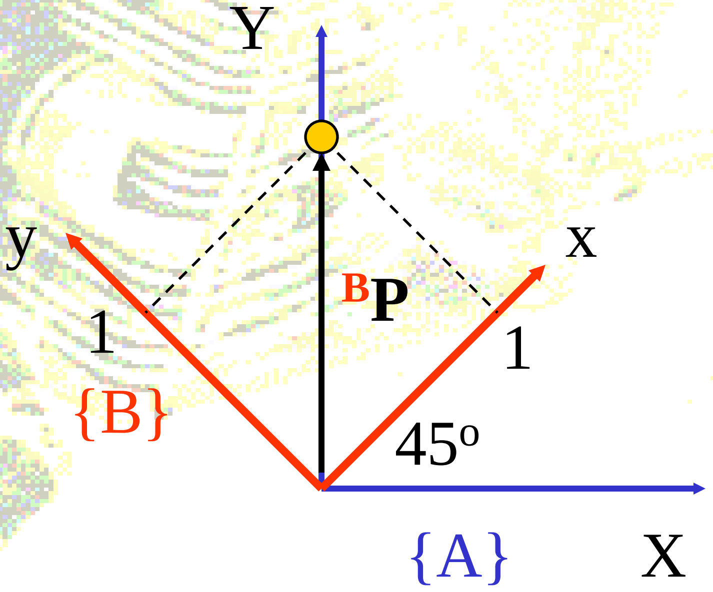
Spatial Descriptions and Transformations

- **Example:**

Given: ^{B}P, Find: ^{A}P

$$^{B}P=\begin{bmatrix}1\\1\\0\end{bmatrix},\quad {}^{A}_{B}R=\begin{bmatrix}0.707 & -0.707 & 0\\0.707 & 0.707 & 0\\0 & 0 & 1\end{bmatrix}$$

$$^{A}P={}^{A}_{B}R\,{}^{B}P=\begin{bmatrix}0\\1.41\\0\end{bmatrix}$$



Spatial Descriptions and Transformations

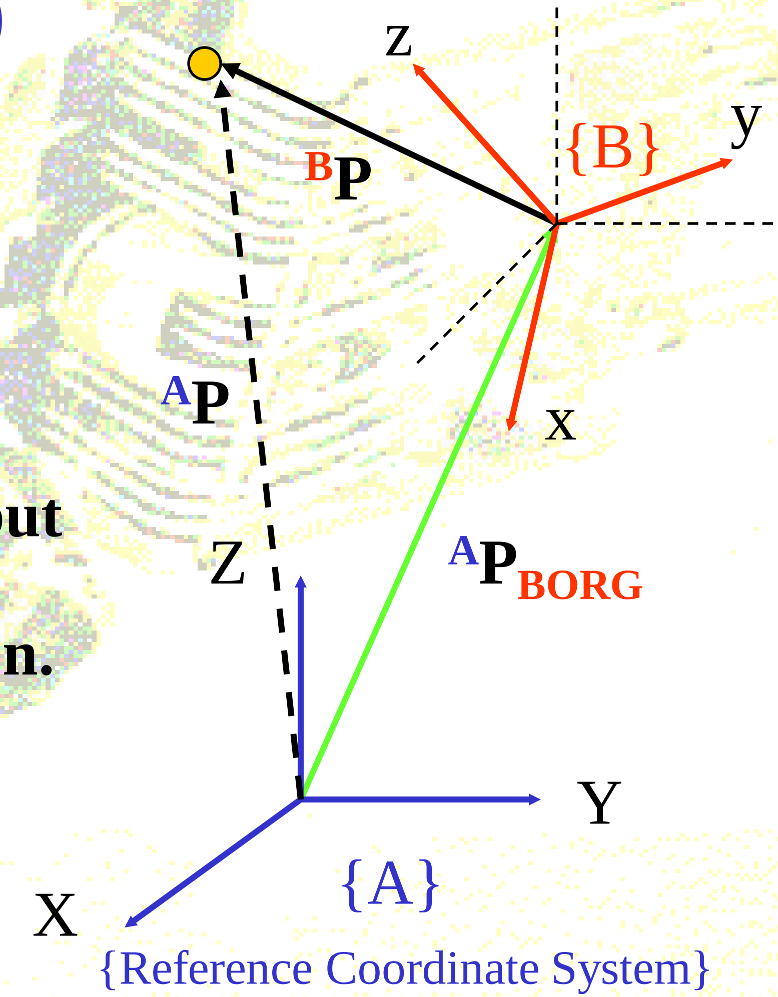
- **Mapping (Changing Relativity)**
  (نگاشت- تغییرنسبیت):

➢ ***Translated & Rotated Frames:***

Given: ^{B}P, Find: ^{A}P

1. **Express ^{B}P with respect to a frame with the same orientation as {A}, but with origin at {B},**
2. **Translation done by vector addition.**

$$^{A}P = {}^{A}_{B}R\,{}^{B}P + {}^{A}P_{BORG}$$



Note: You can only add two vectors when they are expressed in frames with the same orientation.

Spatial Descriptions and Transformations

- **Define a *Transformation Operator* to express mapping in a cleaner form:**

$$\begin{bmatrix} {}^{A}P \\ \hdashline 1 \end{bmatrix} = \left[\begin{array}{c:c} {}^{A}_{B}R_{(3\times3)} & {}^{A}P_{BORG\,(3\times1)} \\ \hdashline 0_{(1\times3)} & 1 \end{array}\right] \begin{bmatrix} {}^{B}P \\ \hdashline 1 \end{bmatrix} \Rightarrow {}^{A}P = {}^{A}_{B}T\,{}^{B}P$$

$${}^{A}_{B}T$$

: ***Homogeneous Transformation Matrix*** **Expressing Position and Orientation of frame {B} relative to frame {A}.**

Transformations?

- Transformations are a way of describing spatial movements/locations
  - Transformations are represented as a frame!
  - Transformations may be "pure"
    - ***rotation*** about a single axis
    - ***translation***
  - Transformations may be a combination of rotation(s) and translation(s)

Position Vector

- A vector that doesn't begin at the origin can be specified by the difference between two points A and B

$$P = (B_x - A_x)i + (B_y - A_y)j + (B_z - A_z)k$$

- A position can also be represented as a **4x1** vector where the 4th number represents a ***scaling factor***

$$P = [x\ y\ z\ w]^{-1}$$

- which is equivalent to $[x/w, y/w, z/w]^{-1}$

Vector Example

A vector ***P*** is given as ***3x + 5y + 2z***. Express the vector as:

1) As a vector with scale factor 2	$P = [6\ 10\ 4\ 2]^{-1}$
2) As a directional vector	$P = [3\ 5\ 2\ 0]^{-1}$
3) As a unit vector	$P = [.48\ .811\ .324\ 1]^{-1}$
4) As a directional unit vector	$P = [.48\ .811\ .324\ 0]^{-1}$

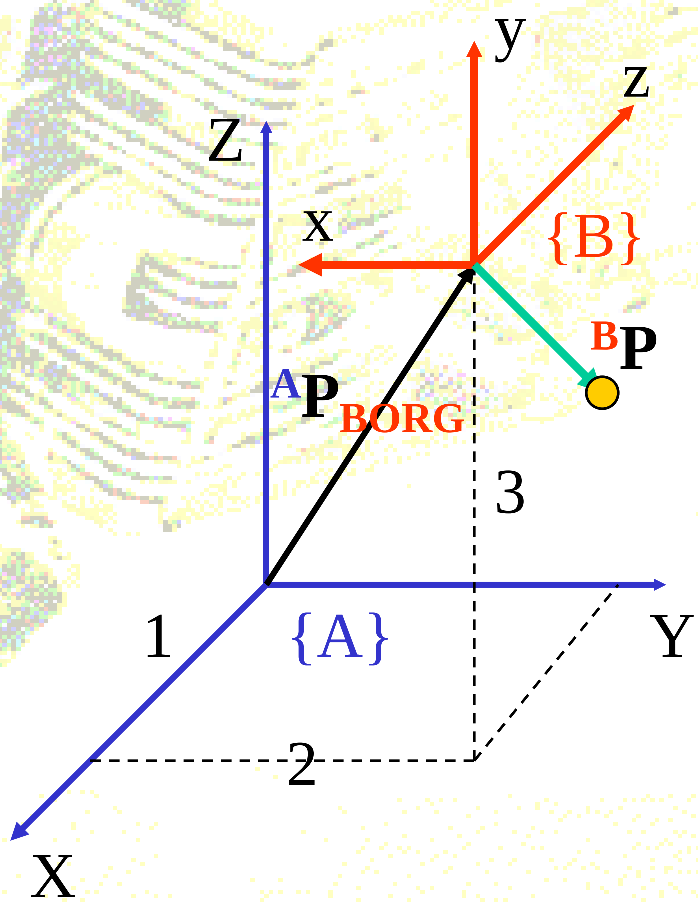
Spatial Descriptions and Transformations

- **Example:**

Given: ^{B}P, Find: ^{A}P

$$^{B}P = \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \quad {}^{A}_{B}T = \begin{bmatrix} 0 & 0 & -1 & 1 \\ -1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^{A}P = {}^{A}_{B}T\,{}^{B}P = \begin{bmatrix} 2 \\ 3 \\ 2 \\ 1 \end{bmatrix}$$

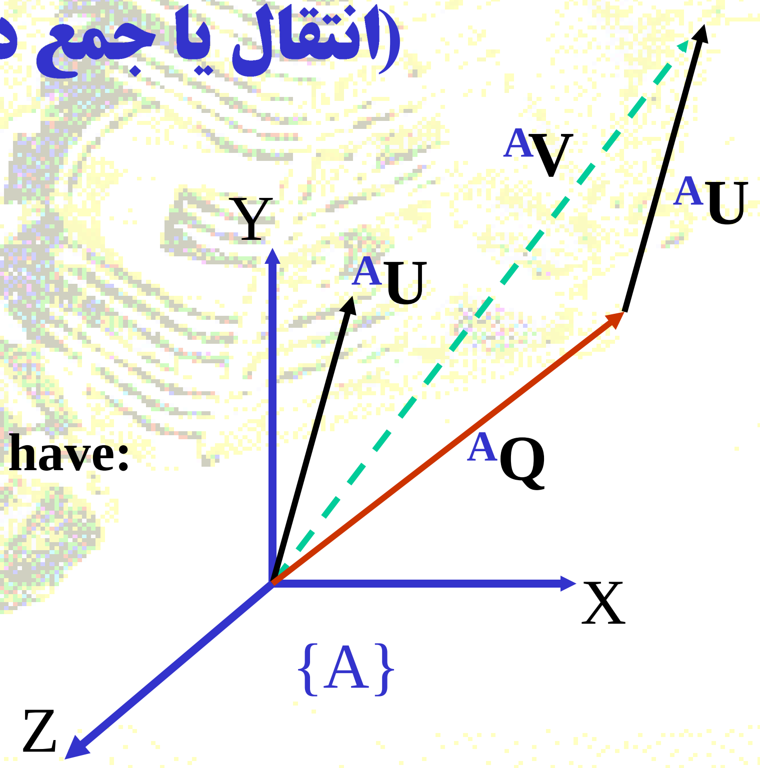


Spatial Descriptions and Transformations

- **Operators:** **Translations, Rotations, Transformations**

➢ **Translation Operator: (انتقال یا جمع دو بردار)**

$$Trans(Q) \equiv Trans(q_x, q_y, q_z) \equiv \begin{bmatrix} 1 & 0 & 0 & q_x \\ 0 & 1 & 0 & q_y \\ 0 & 0 & 1 & q_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Given a vector U=[x,y,z,1]^T and Trans(Q), we have:

$$V = Trans(Q)U = \begin{bmatrix} x + q_x \\ y + q_y \\ z + q_z \\ 1 \end{bmatrix}$$

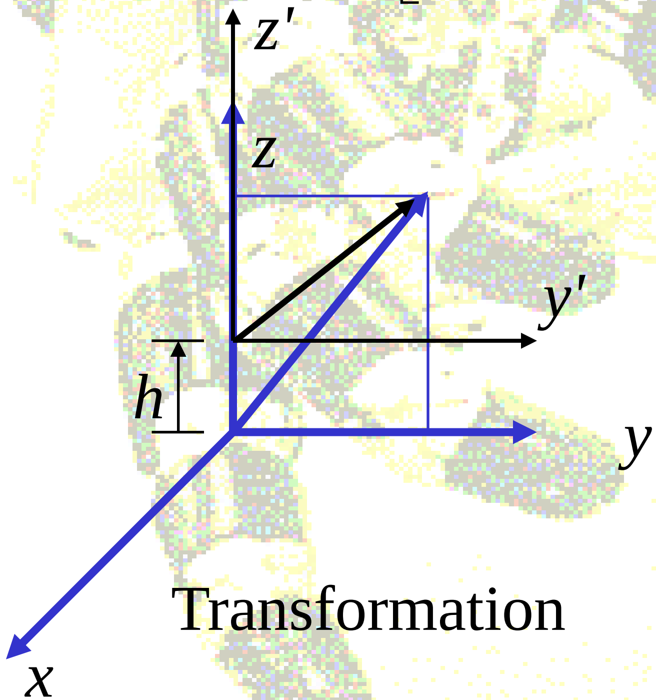
$$V = (x\hat{i} + y\hat{j} + z\hat{k}) + (q_x\hat{i} + q_y\hat{j} + q_z\hat{k})$$

Example: Translation

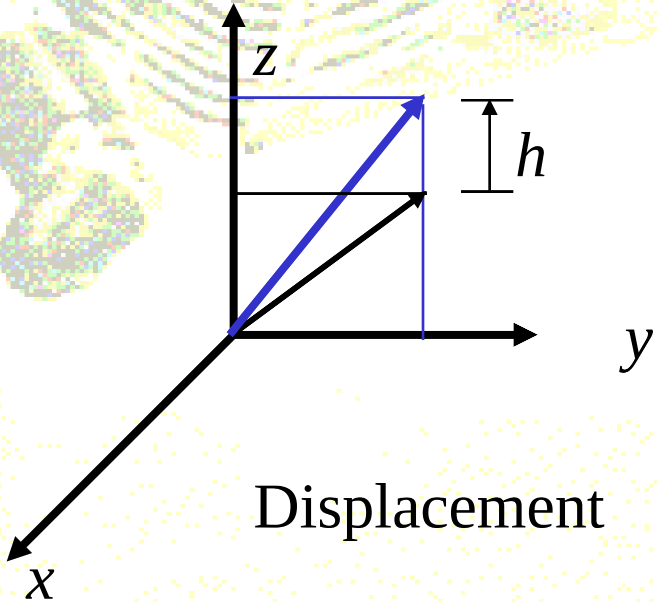
– Translation along the *z*-axis through *h*

$$Trans(z,h)=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&h\\0&0&0&1\end{bmatrix}$$

$$\begin{bmatrix}x\\y\\z\\1\end{bmatrix}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&h\\0&0&0&1\end{bmatrix}\begin{bmatrix}x'\\y'\\z'\\1\end{bmatrix}$$



Transformation



Displacement

Spatial Descriptions and Transformations

- **Operators: Translations, Rotations, Transformations**
  - **Rotation Operator: (دوران یک بردار)**

Rotation Matrix/operator is used to operate on a vector ${}^{A}P_{1}$ and changes that vector to a new vector, ${}^{A}P_{2}$, by means of a rotation Rot(θ).

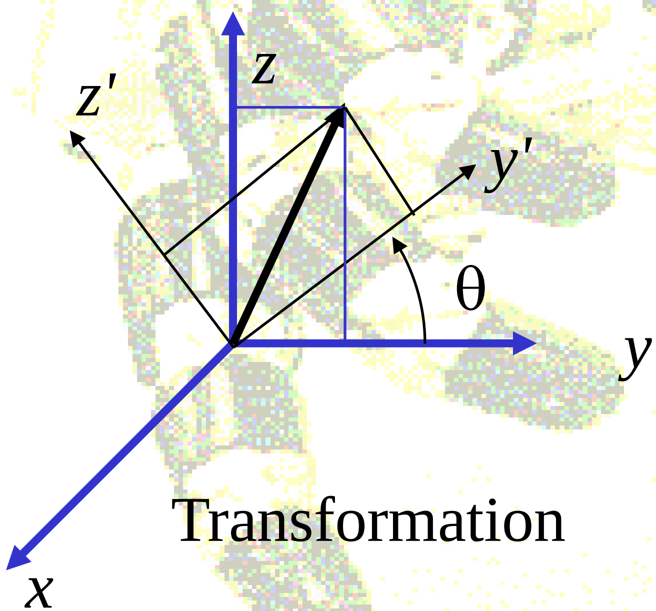
$$Rot(x,\theta)=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & Cos\theta & -Sin\theta & 0\\ 0 & Sin\theta & Cos\theta & 0\\ 0 & 0 & 0 & 1\end{bmatrix}, Rot(y,\theta)=\begin{bmatrix}Cos\theta & 0 & Sin\theta & 0\\ 0 & 1 & 0 & 0\\ -Sin\theta & 0 & Cos\theta & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$Rot(z,\theta)=\begin{bmatrix}Cos\theta & -Sin\theta & 0 & 0\\ Sin\theta & Cos\theta & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

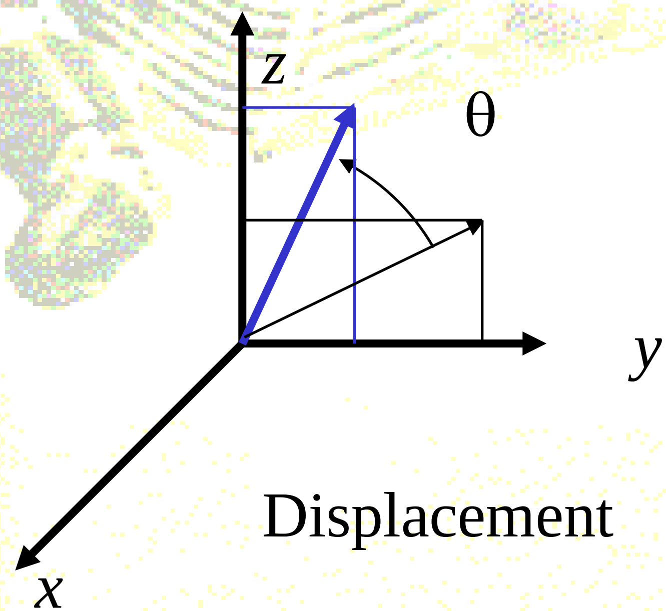
Example: Rotation

– Rotation about the *x*-axis through θ

$$Rot(x,\theta)=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & \cos\theta & -\sin\theta & 0\\ 0 & \sin\theta & \cos\theta & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\qquad \begin{bmatrix}x\\ y\\ z\\ 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & \cos\theta & -\sin\theta & 0\\ 0 & \sin\theta & \cos\theta & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x'\\ y'\\ z'\\ 1\end{bmatrix}$$



Transformation

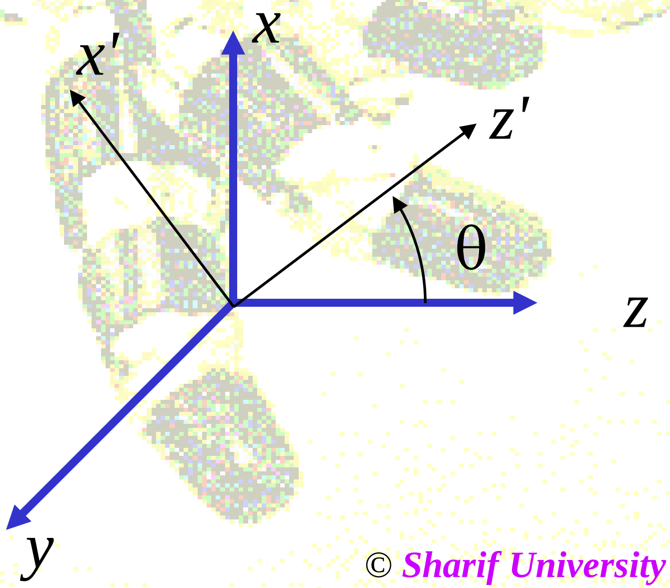


Displacement

Example: Rotation

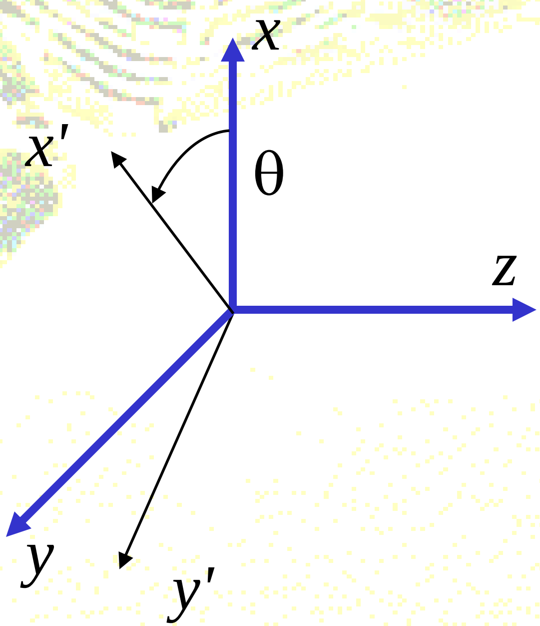
- Rotation about the *y*-axis through θ

$$Rot(y,\theta)=\begin{bmatrix}\cos\theta & 0 & \sin\theta & 0\\ 0 & 1 & 0 & 0\\ -\sin\theta & 0 & \cos\theta & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$



- Rotation about the *z*-axis through θ

$$Rot(z,\theta)=\begin{bmatrix}\cos\theta & -\sin\theta & 0 & 0\\ \sin\theta & \cos\theta & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$



Spatial Descriptions and Transformations

- **Operators: Translations, Rotations, Transformations**
  - **Transformation Operator: (اپراتور تبد یل)**

A combination of both Translation and Rotation Operators.

Given: $^{A}P_1$, **Rotate it about Z_A-axis by 30^0, Translate 3" along X_A and 5" along Y_A**

Find: $^{A}P_2 = ?$

$$^{A}P_1 = \begin{bmatrix} 5 \\ 2 \\ 0 \end{bmatrix} \quad , \quad T = \begin{bmatrix} 0.866 & -0.5 & 0 & 3 \\ 0.5 & 0.866 & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^{A}P_2 = T\,^{A}P_1 = \begin{bmatrix} 6.33 \\ 9.23 \\ 0 \\ 1 \end{bmatrix}$$

Chapter 2 Exercises:

- 2.1, 2.3, 2.4, 2.5, 2.11, 2.12, 2.13, 2.16, 2.32, 2.35
- 2.1 Programming Exercise (program Atan2 function)
- 2A MathLab Exercise